CS 511 : Lecture Notes

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Today's lecture : Examples for first order definability of relations and functions

Preface: Always remember to note the distinction between a symbol's colliqually meaning, and it's model interpretation (e.g. given a model $\mathcal{M}$, $+^{\mathcal{M}}$ can be different than $+$)

Example 1:

Suppose the model we're looking at is on the natural numbers with 1 binary predicate $(\mathbb{N}; <)$

Assume that $\doteq$ is always available and interpreted as equality.

Is zero is first order definable in $(\mathbb{N}; <)$? Yes

$$\varphi_{\{0\}}(x) \triangleq \forall y \ (x \doteq y \lor x < y)$$

Take a look at this statement intuitively, for every element in $\mathbb{N}$, either x is that element, or x is less than. In this case 0 is the only elment that will fit this definition. Let's ask ourselves, can we simplify this? (Yes, if we just had $\varphi_{\{0\}}(x) \triangleq \forall y \ (x < y)$, 0 would be the only item to satisfy this statement in this model). Let's check if this formula is correct

$$\begin{aligned} R &= \{a \in \mathbb{N} \mid (\mathbb{N}; <; a) \models \varphi_{\{0\}}\} \\ &= \{a \in \mathbb{N} \mid (\mathbb{N}; <) \models \varphi_{\{0\}}[a]\} \\ &= \{0\} \ \checkmark \end{aligned}$$

More definitions:

$$\begin{aligned} &\varphi_{\{1\}}(x) \triangleq \neg\varphi_{\{0\}}(x) \land \forall y \ (\neg\varphi_{\{0\}}(y) \to (x \doteq y \lor x < y)) \\ &\varphi_{\{2\}}(x) \triangleq \neg\varphi_{\{0\}}(x) \land \neg\varphi_{\{1\}}(x) \land \forall y \ ((\neg\varphi_{\{0\}}(y) \lor \neg\varphi_{\{1\}}(y)) \to (x \doteq y \lor x < y)) \end{aligned}$$

We can easily simplfy some of these formulas, for example in $\varphi_{\{2\}}(x)$ we can remove the $\varphi_{\{0\}}(x)$ in the implication due to the fact that $\varphi_{\{1\}}(x)$ already captures $\varphi_{\{0\}}(x)$ in it's own formula

$$\varphi_{\{2\}}(x) \triangleq \neg\varphi_{\{0\}}(x) \wedge \neg\varphi_{\{1\}(x)} \wedge \forall y\ (\neg\varphi_{\{1\}}(y) \rightarrow (x \doteq y \vee x < y))$$

If we wanted to keep going for higher numbers, $\varphi_{\{n\}}(x)$ becomes annoyingly long. To make things easier let's do the following

We want to define a $\Psi_{\{n_1,\ldots,n_k\}}(x) \triangleq \neg\varphi_{\{n_1\}}(x) \wedge \neg\varphi_{\{n_2\}} \wedge \ldots \wedge \neg\varphi_{\{n_k\}}$. Can we do this?

CLAIM: There is a first-order WFF $\varphi_{\{n\}}(x)$ s.t $R \triangleq \{a \in \mathbb{N} \mid (\mathbb{N}; <) \models \varphi_{\{n\}}[a]\} = \{n\}$

For every finite $X \subseteq \mathbb{N}$, there is a first-order wff $\varphi_X(x)$ which uniquely defines X. Suppose that $X = \{n_1, n_2, \ldots, n_k\} \quad k \geq 1$. Then

$$\varphi_X(x) \triangleq \varphi_{\{n_1\}}(x) \vee \varphi_{\{n_2\}}(x) \vee \ldots \varphi_{\{n_k\}}(x)$$

So yeah, we're good, let $\Psi_X(x) \triangleq \neg\varphi_X(x)$

What about if X was infinite? Impossible to make a first-order WFF, however, if X is cofinite we can define a first-order WFF

$$\Psi_X(x) \triangleq \neg\varphi_{\{n_1\}}(x) \wedge \ldots \wedge \neg\varphi_{\{n_k\}}(x)$$

Where $X = \mathbb{N} \backslash \{n_1, \ldots, n_k\}$

Example 2:

Let $\mathcal{M}$ be a model where we define $(\mathbb{N}; +; 0)$

Then we can define a our normal intutition of $<$ as such

$$\varphi_<(x, y) \triangleq \exists z\ (\neg(z \doteq 0) \wedge (x + z \doteq y))$$

Example 3:

Define monus $\dot{-}$ such that

$$m \dot{-} n = \begin{cases} 0 & m < n \\ m - n & m \geq n \end{cases}$$

Can we use the same model in example 2 to define monus? (Spoilers, yes)

$$\varphi_{\dot{-}}(x, y, z) \triangleq (\varphi_<(x, y) \rightarrow (z \doteq 0)) \wedge (\neg\varphi_<(x, y) \rightarrow x \doteq y + z)$$

Example 4:

Let $\mathcal{M}$ be a model where we define $(\mathbb{N}; |; +; 0)$

Where

$$m|n = \begin{cases} \text{True} & m \text{ is a divisor of n} \\ \text{False} & \text{o.w} \end{cases}$$

Can we define the least common multiple function? $(lcm(m, n) = p)$

$$\varphi_{lcm}(x, y, v) \triangleq (x|v) \wedge (y|v) \wedge \forall w \ \big((x|w) \wedge (y|w) \rightarrow (vw \vee \varphi_{<}(v, w))\big)$$

Questions to answer: Can we simplfy this? Yes. Can we define lcm without using the less than formula we defined earlier? Yes.

More examples of definiability can be found in the slides.

Slide Example:

Say we have $(\mathbb{N}; 0; S)$ where S is the successor function. Can we define addition?

What's wrong with this? (Exercise is left up to the reader)

$$\forall x \forall y \forall z (\underbrace{S \ldots S}_{y} x \doteq z)$$

Fact: addition is NOT first order definable from "0" and "S". More facts in the handout page 14.