Solutions for Mid-Term Exam

Thursday, March 5, 2015 (with adjustments on March 24, 2015)

Problem 1. Modeling with Propositional Logic. There are \( n \geq 1 \) radio stations \( S = \{ s_1, \ldots, s_n \} \). Each station must be assigned one of \( k \geq 1 \) transmission frequencies \( F = \{ f_1, \ldots, f_k \} \). In typical medium-size cities, \( n \) is a much larger number than \( k \), which precludes the possibility of assigning every frequency to only one radio station.

The assignment of frequencies is made on the basis of a set \( E \subseteq S \times S \), whose intended meaning is: if \( (s_i, s_j) \in E \), with \( i \neq j \), then stations \( s_i \) and \( s_j \) are too close and cannot be assigned the same frequency. Your task is to model the preceding situation in order to answer the following question:

Question: Is it possible to assign a transmission frequency to every station such that no station pairs in \( E \) have the same frequency?

You have to formulate a propositional WFF \( \varphi \) whose satisfiability answers the preceding question. Write \( \varphi \) as the conjunction of three subformulas, \( \varphi \triangleq \varphi_1 \land \varphi_2 \land \varphi_3 \), where:

(a) \( \varphi_1 \) should express “every radio station is assigned at least one transmission frequency”:

**Answer:** We can define \( \varphi_1 \) as a conjunction of \( n \) disjunctions, with each disjunction having \( k \) literals:

\[
\varphi_1 \triangleq \bigwedge_{i=1}^{n} \left( \bigvee_{j=1}^{k} x_{i,j} \right)
\]

(b) \( \varphi_2 \) should express “every radio station is assigned not more than one transmission frequency”:

**Answer:** We can define \( \varphi_2 \) as follows:

\[
\varphi_2 \triangleq \bigwedge_{i=1}^{n} \left( \bigwedge_{j=1}^{k-1} \left( x_{i,j} \rightarrow \bigwedge_{j<\ell \leq k} \neg x_{i,\ell} \right) \right)
\]

(c) \( \varphi_3 \) should express “every station pair \( (s_i, s_j) \) in \( E \) are not assigned the same frequency”:

**Answer:** We can define \( \varphi_3 \) as follows:

\[
\varphi_3 \triangleq \bigwedge_{(s_i, s_j) \in E} \left( \bigwedge_{\ell=1}^{k} \left( x_{i,\ell} \rightarrow \neg x_{j,\ell} \right) \right)
\]

**Hint:** Use propositional atom \( x_{i,j} \) to denote that radio station \( s_i \) is assigned frequency \( f_j \), for every \( 1 \leq i \leq n \) and every \( 1 \leq j \leq k \).

Problem 2. Modeling with First-Order Logic. You will show in this problem that every infinite planar graph is four-colorable. The theory of simple undirected graphs can be taken as a set \( \Gamma \) of two axioms over signature \( \Sigma \triangleq \{ R, = \} \) consisting of one binary predicate symbol and the equality symbol:

\[
\Gamma \triangleq \left\{ \forall x, \forall y. R(x, y) \rightarrow R(y, x), \ \forall x. \neg R(x, x) \right\}
\]
Expand the signature $\Sigma$ to $\Sigma' = \Sigma \cup \{B, G, P, Y\}$ where $B$, $G$, $P$, and $Y$ are unary predicate symbols (for 'blue', 'green', 'purple', and 'yellow').

(a) Write a first-order sentence $\varphi_1$ which, in any $\Sigma'$-structure $M$ satisfying $\Gamma$ (i.e., $M$ is a simple undirected graph), asserts “every vertex has at least one of the colors: blue, green, purple, yellow”:

**Answer:**

$$\varphi_1 \triangleq \forall x.(B(x) \lor G(x) \lor P(x) \lor Y(x))$$

(b) Write a first-order sentence $\varphi_2$ which, in any $\Sigma'$-structure $M$ satisfying $\Gamma$, asserts “every vertex has at most one color”:

**Answer:**

$$\varphi_2 \triangleq \forall x. \left( \begin{array}{l} B(x) \rightarrow \neg(G(x) \lor P(x) \lor Y(x)) \\ G(x) \rightarrow \neg(B(x) \lor P(x) \lor Y(x)) \\ P(x) \rightarrow \neg(B(x) \lor G(x) \lor Y(x)) \\ Y(x) \rightarrow \neg(B(x) \lor G(x) \lor P(x)) \end{array} \right)$$

(c) Write a first-order sentence $\varphi_3$ which, in any $\Sigma'$-structure $M$ satisfying $\Gamma$, asserts “no two adjacent vertices have the same color”:

**Answer:**

$$\varphi_3 \triangleq \left( \begin{array}{l} \forall x. \forall y. \neg(B(x) \land B(y) \land R(x, y)) \\ \forall x. \forall y. \neg(G(x) \land G(y) \land R(x, y)) \\ \forall x. \forall y. \neg(P(x) \land P(y) \land R(x, y)) \\ \forall x. \forall y. \neg(Y(x) \land Y(y) \land R(x, y)) \end{array} \right)$$

(d) Show that if $M$ is an infinite planar graph, i.e.,

- $M$ is a $\Sigma$-structure satisfying $\Gamma$,
- the domain of $M$ is infinite, and
- $M$ is planar as a graph,

then there is a $\Sigma'$-structure $M'$, which expands $M$ with four unary relations $B^{M'}$, $G^{M'}$, $P^{M'}$, and $Y^{M'}$, and which satisfies $\varphi_1 \land \varphi_2 \land \varphi_3$, i.e., $M'$ is four-colorable and, thus, $M$ is also four-colorable.
Answer: Let \( \mathcal{M} \triangleq (A, R^\mathcal{M}, =^\mathcal{M}) \) be a \( \Sigma \)-structure, where \( \Sigma \triangleq \{R, =\} \). The signature \( \Sigma' \triangleq \Sigma \cup \{B, G, P, Y\} \) is an expansion of \( \Sigma \) which includes four new unary predicate symbols. Another expanded signature is \( \Sigma_A \) which adds to \( \Sigma \) a constant symbol \( c_a \) for every \( a \in A \):

\[
\Sigma_A \triangleq \Sigma \cup \{c_a \mid a \in A\}.
\]

We write \( \Sigma'_A \) for the expansion of \( \Sigma' \) which adds a constant symbol \( c_a \) for every \( a \in A \):

\[
\Sigma'_A \triangleq \Sigma' \cup \{c_a \mid a \in A\} = \Sigma \cup \{B, G, P, Y\} \cup \{c_a \mid a \in A\}.
\]

Any \( \Sigma'_A \)-structure \( \mathcal{M}'_A \) will therefore have the form:

\[
\mathcal{M}'_A \triangleq (X, R^{\mathcal{M}'_A}, =^{\mathcal{M}'_A}, B^{\mathcal{M}'_A}, G^{\mathcal{M}'_A}, P^{\mathcal{M}'_A}, Y^{\mathcal{M}'_A}, \{c_a^{\mathcal{M}'_A} \mid a \in A\}).
\]

We write \( X \) for the domain of \( \mathcal{M}'_A \) to indicate that \( X \) does not have to be the same as \( A \). We can take a reduct of \( \mathcal{M}'_A \) by omitting all the distinguished constants to obtain:

\[
\mathcal{M}' \triangleq (X, R^{\mathcal{M}'}, =^{\mathcal{M}'}, B^{\mathcal{M}'}, G^{\mathcal{M}'}, P^{\mathcal{M}'}, Y^{\mathcal{M}'}),
\]

where \( R^{\mathcal{M}'} = R^{\mathcal{M}'_A}, B^{\mathcal{M}'} = B^{\mathcal{M}'_A}, G^{\mathcal{M}'} = G^{\mathcal{M}'_A}, P^{\mathcal{M}'} = P^{\mathcal{M}'_A}, \) and \( Y^{\mathcal{M}'} = Y^{\mathcal{M}'_A} \). Yet, another reduct of \( \mathcal{M}'_A \) is obtained by omitting the four unary predicates to obtain:

\[
\mathcal{M}_A \triangleq (X, R^{\mathcal{M}_A}, =^{\mathcal{M}_A}, \{c_a^{\mathcal{M}_A} \mid a \in A\}),
\]

where \( R^{\mathcal{M}_A} = R^{\mathcal{M}'_A} \) and \( c_a^{\mathcal{M}_A} = c_a^{\mathcal{M}'_A} \) for every \( a \in A \). One particular \( \Sigma_A \)-structure is:

\[
\overline{\mathcal{M}}_A \triangleq (A, R^{\overline{\mathcal{M}}_A}, =^{\overline{\mathcal{M}}_A}, \{c_a^{\overline{\mathcal{M}}_A} \mid a \in A\}),
\]

with the same domain \( A \) as \( \mathcal{M} \), and where \( R^{\overline{\mathcal{M}}_A} = R^\mathcal{M} \) and \( c_a^{\overline{\mathcal{M}}_A} = a \). We can therefore also write:

\[
\overline{\mathcal{M}}_A \triangleq (A, R^\mathcal{M}, =^\mathcal{M}, a_0, a_1, a_2, \ldots),
\]

where \( a_0, a_1, a_2, \ldots \) is an enumeration of all elements of domain \( A \), \( i.e., \overline{\mathcal{M}}_A \) is the original structure \( \mathcal{M} \) after making every element of \( A \) a distinguished constant of \( \overline{\mathcal{M}}_A \). Consider now the set of sentences:

\[
\Delta_A \triangleq \left\{\neg(c_a = c_b) \mid a, b \in A \text{ and } a \neq b\right\} \cup \left\{\psi \mid \psi \text{ is a quantifier-free sentence s.t. } \overline{\mathcal{M}}_A \models \psi\right\}.
\]

One model of \( \Delta_A \) is \( \overline{\mathcal{M}}_A \), but there are infinitely many other models of \( \Delta_A \) such that the original \( \mathcal{M} \) is embedded in each one of them — **convince yourself of this last assertion!** — \( i.e., \) the original \( \mathcal{M} \) is a submodel (here, a subgraph induced by the subset \( A \)) in each.\(^a\) Consider the set of sentences \( \Delta \):

\[
\Delta \triangleq \Gamma \cup \{\varphi_1 \land \varphi_2 \land \varphi_3\} \cup \Delta_A.
\]

To conclude, we show that \( \Delta \) has a model. That \( \Delta \) has a model follows by Compactness and by invoking the fact that every finite planar graph is four-colorable, \( i.e., \) every finite subset of \( \Delta \) has a model.

\(^a\) The definition of \( \Delta_A \) can be simplified, by writing \( \Delta_A \triangleq \{\psi \mid \psi \text{ is a quantifier-free sentence s.t. } \overline{\mathcal{M}}_A \models \psi\} \), because \( \{\neg(c_a = c_b) \mid a, b \in A \text{ and } a \neq b\} \subseteq \{\psi \mid \psi \text{ is a quantifier-free sentence s.t. } \overline{\mathcal{M}}_A \models \psi\} \), I explicitly include in \( \Delta_A \) the set \( \{\neg(c_a = c_b) \mid a, b \in A \text{ and } a \neq b\} \) to stress connections with applications of Compactness presented in lecture.
**Hint 1:** Find a way to make use of the following fact: Every finite planar graph is four-colorable. (Do not try to prove this fact, which is difficult, but you are allowed to invoke it.)

**Hint 2:** If $M$ is a planar graph, then every subgraph of $M$ is also planar. A subgraph of $M$ is a graph whose vertices are a subset of the vertices of $M$ and whose adjacency relation is a subset of the adjacency relation of $M$ restricted to this subset.

**Problem 3. Soundness and Completeness.** Let $\varphi(x,y)$ be a first-order WFF with two free variables $x$ and $y$, and $f$ a unary function symbol not appearing in $\varphi$.

(a) Show that the following sentence is formally provable:

$$\forall x. \varphi(x,f(x)) \rightarrow \forall x. \exists y. \varphi(x,y)$$

**Hint:** First show the following semantic entailment: $\forall x. \varphi(x,f(x)) \models \forall x. \exists y. \varphi(x,y)$, and then complete the argument.

**Answer:** Let $\forall x. \psi \equiv \forall x. \varphi(x,f(x))$ and $\forall x. \exists y. \psi' \equiv \forall x. \exists y. \varphi(x,y)$.

Consider every model $M \models (M, \ldots, f^M, \ldots)$ such that $M \models \forall x. \psi$. In every such model $M$ and for every $a \in M$, it is the case that $M \models \psi[a]$, i.e., $M \models \varphi[a, f^M(a)]$, by the definition of the satisfaction relation “$|$”.

Hence, in every such model $M$ and for every $a \in M$, it is also the case that $M \models (\exists y. \psi')[a]$ by the definition of “$|$”, and that $M \models \forall x. \exists y. \psi'$ by the definition of “$|$” again.

Hence, whenever $M \models \forall x. \psi$ holds, it is the case that $M \models \forall x. \exists y. \psi'$ also holds.

Hence, for all models $M$, we have $M \models (\forall x. \psi) \to (\forall x. \exists y. \psi')$. By completeness of first-order logic, $\vdash (\forall x. \psi) \to (\forall x. \exists y. \psi')$, which is the desired conclusion.

(b) Show that the following sentence is not formally provable:

$$\forall x. \exists y. \varphi(x,y) \rightarrow \forall x. \varphi(x,f(x))$$

**Hint:** First show the following semantic entailment does not hold in general:

$\forall x. \exists y. \varphi(x,y) \models \forall x. \varphi(x,f(x))$. You need to define a counter-example. Then complete the argument.

**Answer:** An appropriate counter-example is to consider the model $\mathcal{N} \equiv (\mathbb{N}, S, \cdot)$ where “$S$” is the (unary) successor operation and “$\cdot$” is the (binary) multiplication operation on natural numbers. Let $\varphi(x,y) \equiv (x \cdot x = y)$.

By the standard properties of natural numbers, we have $\mathcal{N} \models \forall x \exists y \varphi(x,y)$, which expresses the fact that for every natural number $a$ there is a natural number $b$ such that $b$ is equal to the square of $a$.

On the other hand, $\mathcal{N} \not\models \forall x \varphi(x,f(x))$ where $f^\mathcal{N} = S$, which expresses the fact that “it is not the case that, for every natural number $a$, the successor of $a$ is equal to the square of $a$”.

Hence, for every model $M$ with the appropriate signature, $M \not\models \forall x \exists y \varphi(x,y) \to \forall x \varphi(x,f(x))$. Hence, by soundness of first-order logic, $\not\vdash \forall x \exists y \varphi(x,y) \to \forall x \varphi(x,f(x))$, which is the desired conclusion.

**Problem 4. First-Order Definability.** Consider the relational structure (or model):

$\mathcal{N} = (\mathbb{N}; =, \cdot, S, +, 0)$ where

- $\mathbb{N} = \{0, 1, 2, \ldots\}$, the set of natural numbers,
- $|$ is a binary predicate, written in infix, such that $x|y = true$ iff “$x$ is a divisor of $y$”,
- $\cdot$ is a binary function symbol, written in infix, such that $x \cdot y = \text{the product of } x \text{ and } y$.

Consider the relational structure (or model):
Write an appropriate first-order WFF to show each of the following:

(a) The order predicate $\lt : \mathbb{N} \times \mathbb{N} \to \{\text{true}, \text{false}\}$ is first-order definable in $(\mathbb{N}; =, +, 0)$.

**Answer:** “$x < y$” is first-order definable in $(\mathbb{N}; =, +, 0)$ by:

$$\varphi_<(x, y) \triangleq \exists z ((z = 0) \land (x + z = y))$$

(b) The subtraction operation $- : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is first-order definable in $(\mathbb{N}; =, +, 0)$.

**Hint:** If $\varphi(x, y, z)$ is supposed to first-order define “$x - y = z$”, it is not enough to write $\varphi(x, y, z) \triangleq (x = y + z)$, because if $x < y$ then $x - y = 0$.

**Answer:** “$x - y = z$” is first-order definable in $(\mathbb{N}; =, +, 0)$ by:

$$\varphi_-(x, y, z) \triangleq (\varphi_<(x, y) \to z = 0) \land (\neg \varphi_<(x, y) \to x = y + z)$$

(c) The operation $\text{lcm} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (least common multiple) is first-order definable in $(\mathbb{N}; =, |, S, +, 0)$.

**Answer:** “$\text{lcm}(x, y) = v$” is first-order definable in $(\mathbb{N}; =, |, S, +, 0)$ by:

$$\varphi_{\text{lcm}}(x, y, v) \triangleq (x | v) \land (y | v) \land \forall w \left( (x | w) \land (y | w) \to (v = w \lor \varphi_<(v, w)) \right)$$

A somewhat shorter definition of $\varphi_{\text{lcm}}(x, y, v)$ is the following:

$$\varphi'_{\text{lcm}}(x, y, v) \triangleq \forall w \left( (x | w) \land (y | w) \leftrightarrow (v | w) \right)$$

**Problem 5. Second-Order Definability.** The following sentence (closed WFF) $\varphi$ is second-order:

$$\varphi \triangleq \exists F \left[ \forall x \forall y (F(x) = F(y) \to x = y) \land \exists z \forall x \neg (F(x) = z) \right]$$

$\varphi$ is written using one unary-function variable $F$.

(a) Which of the following does $\varphi$ express (check the appropriate box):

- $\Box$ $F$ is a bijection, i.e., it is both injective and surjective.
- $\Box$ $F$ is an injective, but not surjective.
- $\Box$ $F$ is a surjective, but not injective.
- $\Box$ $F$ is a permutation of the domain of the model.

(b) Suppose for some model $\mathcal{M}$ we have $\mathcal{M} \models \varphi$. Prove or disprove that $\mathcal{M}$ must be an infinite model.
Note: The proof you have to write for (b) is at the meta-level. Even though it is not a formal proof, it must still be a rigorous proof.

**Answer:** From a course on Discrete Mathematics, if a unary function \( f : A \to A \) on a finite domain \( A \) is injective (i.e., one-one), then it is also surjective (i.e., onto).

You can say more, in fact. If \( A \) is finite, then a unary function \( f : A \to A \) is injective iff it is surjective. If a unary function on \( A \) is both injective and surjective, then it is a bijection (i.e., a permutation).

The second-order sentence \( \varphi \) asserts the existence of a unary function \( F \) which is injective but not surjective. Such a unary function can exist only on infinite domains. Hence, if a model \( M \) is such that \( M \models \varphi \), the domain of \( M \) must be infinite.