An algebraic structure $\mathcal{A}$, or just an algebra $\mathcal{A}$, is a set $A$, called the carrier set or underlying set of $\mathcal{A}$, with one or more operations on the carrier $A$. (Search the Web, here and here, for more details.)
Algebraic Structures: definitions and examples

- An algebraic structure $\mathcal{A}$, or just an algebra $\mathcal{A}$, is a set $A$, called the carrier set or underlying set of $\mathcal{A}$, with one or more operations on the carrier $A$. (Search the Web, here and here, for more details.)

- Examples of algebraic structures:
  - $(\mathbb{Z}, +, \cdot)$
    the set of integers with binary operations addition “$+$” and multiplication “$\cdot$”,
  - $(\mathbb{N}, \text{succ, pred, 0, 1})$
    the set of natural numbers with unary operations, “succ” and “pred”,
    and nullary operations, “0” and “1”,

Assaf Kfoury, CS 512, Spring 2015, Handout 06
An **algebraic structure** $\mathcal{A}$, or just an **algebra** $\mathcal{A}$, is a set $A$, called the **carrier set** or **underlying set** of $\mathcal{A}$, with one or more operations on the carrier $A$. (Search the Web, here and here, for more details.)

Examples of **algebraic structures**:

- $(\mathbb{Z}, +, \cdot)$
  the set of integers with **binary** operations addition “$+$” and multiplication “$\cdot$”,

- $(\mathbb{N}, \text{succ}, \text{pred}, 0, 1)$
  the set of natural numbers with **unary** operations, “succ” and “pred”, and **nullary** operations, “0” and “1”,

- $(T, \text{node}, \text{Lt}, \text{Rt})$ where $T$ is the least set such that:

$$T \supseteq \{a, b, c\} \cup \{\langle t_1, t_2 \rangle \mid t_1, t_2 \in T\}$$

with one **binary** operation “node” and two **unary** operations “Lt” and “Rt”, defined by:
Algebraic Structures: definitions and examples

\begin{align*}
\text{node} : & \ T \times T \to T \quad \text{such that} \quad \text{node}(t_1, t_2) = \langle t_1, t_2 \rangle \\
\text{Lt} : & \ T \to T \quad \text{such that} \quad \text{Lt}(t) = \begin{cases} 
  t_1 & \text{if } t = \langle t_1, t_2 \rangle, \\
  \text{undefined} & \text{otherwise.}
\end{cases} \\
\text{Rt} : & \ T \to T \quad \text{such that} \quad \text{Rt}(t) = \begin{cases} 
  t_2 & \text{if } t = \langle t_1, t_2 \rangle, \\
  \text{undefined} & \text{otherwise.}
\end{cases}
\end{align*}
node : \( T \times T \rightarrow T \) such that \( \text{node}(t_1, t_2) = \langle t_1, t_2 \rangle \)

\( \text{Lt} : T \rightarrow T \) such that \( \text{Lt}(t) = \begin{cases} t_1 & \text{if } t = \langle t_1, t_2 \rangle, \\ \text{undefined} & \text{otherwise}. \end{cases} \)

\( \text{Rt} : T \rightarrow T \) such that \( \text{Rt}(t) = \begin{cases} t_2 & \text{if } t = \langle t_1, t_2 \rangle, \\ \text{undefined} & \text{otherwise}. \end{cases} \)

- Sometimes an **algebraic structure** includes two (or more) carriers, together with operations between them, in which case we say the algebraic structure is **two-sorted** (or **multi-sorted**).
node : \(T \times T \rightarrow T\) such that node\((t_1, t_2) = \langle t_1, t_2 \rangle\)

\(Lt : T \rightarrow T\) such that \(Lt(t) = \begin{cases} t_1 & \text{if } t = \langle t_1, t_2 \rangle, \\ \text{undefined} & \text{otherwise}. \end{cases}\)

\(Rt : T \rightarrow T\) such that \(Rt(t) = \begin{cases} t_2 & \text{if } t = \langle t_1, t_2 \rangle, \\ \text{undefined} & \text{otherwise}. \end{cases}\)

> Sometimes an algebraic structure includes two (or more) carriers, together with operations between them, in which case we say the algebraic structure is two-sorted (or multi-sorted).

> Examples of two-sorted algebraic structures:

> \((\mathbb{Z}, \mathbb{B}, \leq, +, \cdot)\) where \(\mathbb{B} = \{F, T\}\) and \(\leq : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{B}\).
Algebraic Structures: definitions and examples

\[
\text{node} : T \times T \to T \quad \text{such that} \quad \text{node}(t_1, t_2) = \langle t_1, t_2 \rangle
\]

\[
\text{Lt} : T \to T \quad \text{such that} \quad \text{Lt}(t) = \begin{cases} t_1 & \text{if } t = \langle t_1, t_2 \rangle, \\ \text{undefined} & \text{otherwise}. \end{cases}
\]

\[
\text{Rt} : T \to T \quad \text{such that} \quad \text{Rt}(t) = \begin{cases} t_2 & \text{if } t = \langle t_1, t_2 \rangle, \\ \text{undefined} & \text{otherwise}. \end{cases}
\]

- Sometimes an \textit{algebraic structure} includes two (or more) carriers, together with operations between them, in which case we say the algebraic structure is \textit{two-sorted} (or \textit{multi-sorted}).

- Examples of \textit{two-sorted algebraic structures}:
  - \((\mathbb{Z}, \mathbb{B}, \leq, +, \cdot)\) where \(\mathbb{B} = \{F, T\}\) and \(\leq : \mathbb{Z} \times \mathbb{Z} \to \mathbb{B}\).
  - \((T, \mathbb{N}, \text{node}, \text{Lt}, \text{Rt}, | |, \text{height})\) where \(T\) is defined on the previous slide, with \(| | : T \to \mathbb{N}\) and \(\text{height} : T \to \mathbb{N}\).
Sometimes in a **multi-sorted algebraic structure**, such as $(\mathbb{Z}, \mathbb{B}, \leq, +, \cdot)$, we omit the Boolean carrier $\mathbb{B}$ for brevity and simply write $(\mathbb{Z}, \leq, +, \cdot)$.
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This assumes that it is clear to the reader that “$\leq$” is a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{B}$, *i.e.*, “$\leq$” is a binary **relation** (rather than a binary **function** or **operation**). As a binary relation, we can write: $\leq \subseteq \mathbb{Z} \times \mathbb{Z}$.
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This assumes that it is clear to the reader that “$\leq$” is a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{B}$, i.e., “$\leq$” is a binary relation (rather than a binary function or operation). As a binary relation, we can write: $\leq \subseteq \mathbb{Z} \times \mathbb{Z}$.

Strictly speaking, a structure such as $(\mathbb{Z}, \leq, +, \cdot)$, which now includes operations as well as relations, is called a relational structure rather than just an algebraic structure.
Sometimes in a multi-sorted algebraic structure, such as \((\mathbb{Z}, \mathbb{B}, \leq, +, \cdot)\), we omit the Boolean carrier \(\mathbb{B}\) for brevity and simply write \((\mathbb{Z}, \leq, +, \cdot)\).

This assumes that it is clear to the reader that “\(\leq\)” is a function from \(\mathbb{Z} \times \mathbb{Z}\) to \(\mathbb{B}\), i.e., “\(\leq\)” is a binary relation (rather than a binary function or operation). As a binary relation, we can write: 
\[\leq \subseteq \mathbb{Z} \times \mathbb{Z}.\]

Strictly speaking, a structure such as \((\mathbb{Z}, \leq, +, \cdot)\), which now includes operations as well as relations, is called a relational structure rather than just an algebraic structure.

But the transition from algebraic structures to more general relational structures is not demarcated sharply.

In particular, if a structure \(\mathcal{A}\) includes one or two relations with standard meanings (such as “\(\leq\)”), we can continue to call \(\mathcal{A}\) an algebraic structure.
A **partially ordered set**, or **poset** for short, is a set $P$ with a **partial ordering** $\leq$ on $P$, i.e., for all $a, b, c \in P$, the ordering $\leq$ satisfies:

- $a \leq a$ “$\leq$ is reflexive”
- $(a \leq b \text{ and } b \leq a)$ imply $a = b$ “$\leq$ is anti-symmetric”
- $(a \leq b \text{ and } b \leq c)$ imply $a \leq c$ “$\leq$ is transitive”

The ordering $\leq$ is **total** if it also satisfies for all $a, b \in P$:

$(a \leq b) \text{ or } (b \leq a)$
A **partially ordered set**, or **poset** for short, is a set $P$ with a **partial ordering** $\trianglelefteq$ on $P$, i.e., for all $a, b, c \in P$, the ordering $\trianglelefteq$ satisfies:

- $a \trianglelefteq a$ ("$\trianglelefteq$ is reflexive")
- $(a \trianglelefteq b$ and $b \trianglelefteq a)$ imply $a = b$ ("$\trianglelefteq$ is anti-symmetric")
- $(a \trianglelefteq b$ and $b \trianglelefteq c)$ imply $a \trianglelefteq c$ ("$\trianglelefteq$ is transitive")

The ordering $\trianglelefteq$ is **total** if it also satisfies for all $a, b \in P$:

$(a \trianglelefteq b)$ or $(b \trianglelefteq a)$

**Examples of posets:**

1. $(2^A, \trianglelefteq )$ where $A$ is a non-empty set and $\trianglelefteq$ is $\subseteq$,
2. $(\mathbb{N} - \{0\}, \trianglelefteq )$ where $m \trianglelefteq n$ iff “$m$ divides $n$”,
3. $(\mathbb{N}, \trianglelefteq )$ where $\trianglelefteq$ is the usual ordering $\leq$.

In (1) and (2), $\trianglelefteq$ is **not total**; in (3), $\trianglelefteq$ is **total**.
Lattices: definitions and examples

- An **lattice** $\mathcal{L}$ is an algebraic structure $(L, \sqsubseteq, \lor, \land)$ where $\lor$ and $\land$ are **binary operations**, and $\sqsubseteq$ is a **binary relation**, such that:
  - $(L, \sqsubseteq)$ is a poset,
  - for all $a, b \in L$, the **least upper bound** of $a$ and $b$ in the ordering $\sqsubseteq$
    - exists,
    - is unique,
    - and is the result of the operation “$a \lor b$”,
  - for all $a, b \in L$, the **greatest lower bound** of $a$ and $b$ in $\sqsubseteq$
    - exists,
    - is unique,
    - and is the result of the operation “$a \land b$”.

- Examples of lattices:
  - $(2^A, \subseteq, \cup, \cap)$ where $\subseteq$ is $\subseteq$, $\lor$ is $\cup$, and $\land$ is $\cap$
  - $(\mathbb{N} - \{0\}, \sqsubset, \lor, \land)$ where $m \sqsubset n$ iff “$m$ divides $n$”, $\lor$ is “lcm”, and $\land$ is “gcd”.
Lattices: definitions and examples

▶ An **lattice** \( \mathcal{L} \) is an algebraic structure \((L, \sqsubseteq, \lor, \land)\) where \( \lor \) and \( \land \) are **binary operations**, and \( \sqsubseteq \) is a **binary relation**, such that:

▶ \((L, \sqsubseteq)\) is a poset,
▶ for all \( a, b \in L \), the **least upper bound** of \( a \) and \( b \) in the ordering \( \sqsubseteq \)
  ▶ exists,
  ▶ is unique,
  ▶ and is the result of the operation “\( a \lor b \)”,
▶ for all \( a, b \in L \), the **greatest lower bound** of \( a \) and \( b \) in \( \sqsubseteq \)
  ▶ exists,
  ▶ is unique,
  ▶ and is the result of the operation “\( a \land b \)”.  

▶ **Examples of lattices:**

▶ \((2^A, \sqsubseteq, \lor, \land)\) where \( \sqsubseteq \) is \( \subseteq \), \( \lor \) is \( \cup \), \( \land \) is \( \cap \)
Lattices: definitions and examples

- An **lattice** $\mathcal{L}$ is an algebraic structure $(L, \sqsubseteq, \lor, \land)$ where $\lor$ and $\land$ are **binary operations**, and $\sqsubseteq$ is a **binary relation**, such that:
  - $(L, \sqsubseteq)$ is a poset,
  - for all $a, b \in L$, the **least upper bound** of $a$ and $b$ in the ordering $\sqsubseteq$ exists, is unique, and is the result of the operation “$a \lor b$”,
  - for all $a, b \in L$, the **greatest lower bound** of $a$ and $b$ in $\sqsubseteq$ exists, is unique, and is the result of the operation “$a \land b$”.

- **Examples of lattices:**
  - $(2^A, \subseteq, \lor, \land)$ where $\subseteq$ is $\subseteq$, $\lor$ is $\cup$, $\land$ is $\cap$
  - $(\mathbb{N} - \{0\}, \preceq, \lor, \land)$ where $m \preceq n$ iff “$m$ divides $n$”, $\lor$ is “lcm”, $\land$ is “gcd”
A lattice \( L = (\mathcal{L}, \leq, \lor, \land) \) is a distributive lattice if for all \( a, b, c \in \mathcal{L} \), the following equations – also called axioms or equational axioms – are satisfied:

\[
\begin{align*}
    a \land (b \lor c) &= (a \land b) \lor (a \land c) & \text{“\land” distributes over “\lor”} \\
    a \lor (b \land c) &= (a \lor b) \land (a \lor c) & \text{“\lor” distributes over “\land”}
\end{align*}
\]
A lattice \( \mathcal{L} = (L, \sqsubseteq, \lor, \land) \) is a **distributive lattice** if for all \( a, b, c \in L \), the following equations – also called **axioms** or **equational axioms** – are satisfied:

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    a \land (b \lor c) &= (a \land b) \lor (a \land c) \quad \text{“}\land\text{” distributes over “}\lor\text{”} \\
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\end{align*}
\]

**Example of a distributive lattice:**

\((2^A, \subseteq, \cup, \cap)\)
Distributive Lattices: definitions and examples

▶ A lattice $\mathcal{L} = (L, \leq, \lor, \land)$ is a **distributive lattice** if for all $a, b, c \in L$, the following **equations** – also called **axioms** or **equational axioms** – are satisfied:

$$a \land (b \lor c) = (a \land b) \lor (a \land c) \quad \text{“\land” distributes over “\lor”}$$

$$a \lor (b \land c) = (a \lor b) \land (a \lor c) \quad \text{“\lor” distributes over “\land”}$$

▶ Example of a **distributive lattice**:

$$(2^A, \subseteq, \cup, \cap)$$

▶ Is the following an example of a **distributive lattice**?

$$(\mathbb{N} - \{0\}, \ldots \text{ divides } \ldots, \text{lcm}, \text{gcd})$$

▶ For more details on **posets** and **lattices**, go to the Web: [here](#) (Hasse diagrams), [here](#) (distributive lattices), and [here](#).
A bounded lattice is an algebraic structure of the form
\[ \mathcal{L} = (L, \preceq, \lor, \land, \bot, \top) \]
where \( \bot \) and \( \top \) are nullary (or 0-ary) operations on \( L \) (or, equivalently, elements in \( L \)) such that:

1. \( \mathcal{L} = (L, \preceq, \lor, \land) \) is a lattice,
2. \( \bot \preceq a \) or, equivalently, \( \bot \land a = \bot \) for every \( a \in L \),
3. \( a \preceq \top \) or, equivalently, \( a \lor \top = \top \) for every \( a \in L \).

The elements \( \bot \) and \( \top \) are uniquely defined. \( \bot \) is the minimum element, and \( \top \) is the maximum element, of the bounded lattice.
A **bounded lattice** is an algebraic structure of the form

\[ \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top) \]

where \( \bot \) and \( \top \) are **nullary** (or **0-ary**) **operations** on \( L \) (or, equivalently, **elements** in \( L \)) such that:

1. \( \mathcal{L} = (L, \sqsubseteq, \lor, \land) \) is a lattice,
2. \( \bot \sqsubseteq a \) or, equivalently, \( \bot \land a = \bot \) for every \( a \in L \),
3. \( a \sqsubseteq \top \) or, equivalently, \( a \lor \top = \top \) for every \( a \in L \).

The elements \( \bot \) and \( \top \) are uniquely defined. \( \bot \) is the **minimum** element, and \( \top \) is the **maximum** element, of the bounded lattice.

**Example of a bounded lattice:** \((2^A, \subseteq, \cup, \cap, \emptyset, A)\)
Bounded Lattices: definitions and examples

- A **bounded lattice** is an algebraic structure of the form

\[ \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top) \]

where \( \bot \) and \( \top \) are **nullary** (or **0-ary**) **operations** on \( L \) (or, equivalently, **elements** in \( L \)) such that:

1. \( \mathcal{L} = (L, \sqsubseteq, \lor, \land) \) is a lattice,
2. \( \bot \sqsubseteq a \) or, equivalently, \( \bot \land a = \bot \) for every \( a \in L \),
3. \( a \sqsubseteq \top \) or, equivalently, \( a \lor \top = \top \) for every \( a \in L \).

The elements \( \bot \) and \( \top \) are uniquely defined. \( \bot \) is the **minimum** element, and \( \top \) is the **maximum** element, of the bounded lattice.

- Example of a **bounded lattice**: \( (2^A, \subseteq, \cup, \cap, \emptyset, A) \)

- Example a **lattice** with a minimum, but **no** maximum:

\[ (\mathbb{N} - \{0\}, \text{"\_ divides \_"}, \text{lcm}, \text{gcd}, 1) \]
Let $\mathcal{L} = (L, \leq, \lor, \land, \bot, \top)$ be a bounded lattice. An element $a \in L$ has a complement $b \in L$ iff:

$$a \land b = \bot \quad \text{and} \quad a \lor b = \top$$

**FACT**: In a bounded distributive lattice, complements are uniquely defined, i.e., an element $a \in L$ cannot have more than one complement $b \in L$.

**Proof**: Exercise.
Complemented Lattices: definitions and examples

A complemented lattice is a bounded distributive lattice
\( \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top) \) where every element has a complement.

Example of a complemented lattice: \((2^A, \subseteq, \cup, \cap, \emptyset, A)\)

Again, for more details various kinds of lattices, go to the Web: here (Hasse diagrams), here (distributive lattices), here (lattices).
Complemented Lattices: definitions and examples

- A **complemented lattice** is a **bounded distributive lattice** \( \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top) \) where every element has a complement.

- Example of a **complemented lattice**: \((2^A, \subseteq, \cup, \cap, \emptyset, A)\)

- Again, for more details various kinds of **lattices**, go to the Web: [here](#) (Hasse diagrams), [here](#) (distributive lattices), [here](#) (lattices).
A complemented lattice $\mathcal{L} = (L, \preceq, \lor, \land, \bot, \top)$ is almost a Boolean algebra, but not quite!

What is missing is an additional operation on $L$ to map an element $a \in L$ to its complement.
A complemented lattice $L = (L, \leq, \lor, \land, \bot, \top)$ is almost a Boolean algebra, but not quite!

What is missing is an additional operation on $L$ to map an element $a \in L$ to its complement.

A first definition of a Boolean algebra:

$$L = (L, \leq, \lor, \land, \bot, \top, \neg)$$

where:
Boolean Algebras: definitions and examples

- A complemented lattice $\mathcal{L} = (L, \leq, \lor, \land, \bot, \top)$ is almost a Boolean algebra, but not quite!

  What is missing is an additional operation on $L$ to map an element $a \in L$ to its complement.

- A first definition of a Boolean algebra:

  $\mathcal{L} = (L, \leq, \lor, \land, \bot, \top, \neg)$

  where:

  1. $\mathcal{L} = (L, \leq, \lor, \land, \bot, \top)$ is a complemented lattice,

  2. The new operation $\neg$ is unary and maps every $a \in L$ to its complement, i.e.:

     $a \land (\neg a) = \bot$ and $a \lor (\neg a) = \top$
A second definition of a **Boolean algebra**
(easier to compare with Heyting algebras later):

\[ \mathcal{L} = (L, \sqsubseteq, \lor, \land, \perp, \top, \rightarrow) \]

where:
A second definition of a **Boolean algebra**

(easier to compare with Heyting algebras later)

\[ \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top, \rightarrow) \]

where:

1. \( \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top) \) is a **complemented lattice**,  

2. The new operation “\( \rightarrow \)” is **binary** such that \( (a \rightarrow \bot) \) is the complement of \( a \), for every every \( a \in L \).
Boolean Algebras: definitions and examples

- A second definition of a **Boolean algebra** (easier to compare with Heyting algebras later):

  \[ \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top, \rightarrow) \]

  where:
  1. \( \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top) \) is a **complemented lattice**,  
  2. The new operation \( \rightarrow \) is **binary** such that \( (a \rightarrow \bot) \) is the complement of \( a \), for every \( a \in L \).

**FACT**: The two preceding definitions of **Boolean algebras** are equivalent because we can define \( \rightarrow \) in terms of \( \{\lor, \neg\} \):

\[ a \rightarrow b := (\neg a) \lor b \]

as well as define \( \neg \) in terms of \( \{\rightarrow, \bot\} \):

\[ \neg a := a \rightarrow \bot \]
Examples of **Boolean algebras**:

For an arbitrary non-empty set $A$:

$$(\mathcal{P}(A), \subseteq, \cup, \cap, \emptyset, A, \overline{\phantom{X}})$$

where $\overline{X} = A - X$ for every $X \subseteq A$. 

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Assaf Kfoury, CS 512, Spring 2015, Handout 06
Examples of **Boolean algebras**:

- For an arbitrary non-empty set $A$:
  $$(2^A, \subseteq, \cup, \cap, \emptyset, A, \overline{\cdot})$$
  where $\overline{X} = A - X$ for every $X \subseteq A$.

- The standard 2-element Boolean algebra:
  $$(\{0, 1\}, \leq, \lor, \land, 0, 1, \neg)$$
  or $$(\{0, 1\}, \leq, \lor, \land, 0, 1, \rightarrow)$$
  where we write “0” for $\text{F}$ and “1” for $\text{T}$.
Heyting Algebras: definitions and examples

- A **Heyting algebra** is an algebraic structure of the form

\[ \mathcal{L} = (L, \leq, \vee, \wedge, \bot, \top, \rightarrow) \]

where:

- A Heyting algebra is an algebraic structure of the form

\[ \mathcal{L} = (L, \leq, \vee, \wedge, \bot, \top, \rightarrow) \]

FACT: The preceding equations uniquely define the operation \( \rightarrow \).

Proof. Exercise.

FACT: Every Boolean algebra is a Heyting algebra.

Proof. Exercise.
A **Heyting algebra** is an algebraic structure of the form

\[ \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top, \rightarrow) \]

where:

- \( \mathcal{L} = (L, \sqsubseteq, \lor, \land, \bot, \top) \) is a **bounded lattice** – **not** necessarily a **complemented lattice**,
Heyting Algebras: definitions and examples

A Heyting algebra is an algebraic structure of the form

\[ \mathcal{L} = (L, \leq, \lor, \land, \bot, \top, \rightarrow) \]

where:

\[ \mathcal{L} = (L, \leq, \lor, \land, \bot, \top) \] is a bounded lattice – not necessarily a complemented lattice,

The new operation “→” is binary and satisfies the equations:

1. \( a \rightarrow a = \top \)
2. \( a \land (a \rightarrow b) = a \land b \)
3. \( a \rightarrow (b \land c) = (a \rightarrow b) \land (a \rightarrow c) \)
4. \( b \leq a \rightarrow b \)
Heyting Algebras: definitions and examples

- A **Heyting algebra** is an algebraic structure of the form

\[ L = (L, \leq, \lor, \land, \bot, \top, \rightarrow) \]

where:

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- The new operation “→” is **binary** and satisfies the **equations**:

1. \( a \rightarrow a = \top \)
2. \( a \land (a \rightarrow b) = a \land b \)
3. \( a \rightarrow (b \land c) = (a \rightarrow b) \land (a \rightarrow c) \)
4. \( b \leq a \rightarrow b \)

**FACT:** The preceding equations uniquely define the operation “→”.

*Proof.* Exercise.
A **Heyting algebra** is an algebraic structure of the form

\[ \mathcal{L} = (L, \leq, \lor, \land, \bot, \top, \rightarrow) \]

where:

- \( \mathcal{L} = (L, \leq, \lor, \land, \bot, \top) \) is a **bounded lattice** – not necessarily a **complemented lattice**,

- The new operation “\( \rightarrow \)” is **binary** and satisfies the **equations**:
  
  1. \( a \rightarrow a = \top \)
  2. \( a \land (a \rightarrow b) = a \land b \)
  3. \( a \rightarrow (b \land c) = (a \rightarrow b) \land (a \rightarrow c) \)
  4. \( b \leq a \rightarrow b \)

**FACT:** The preceding equations uniquely define the operation “\( \rightarrow \)”.  
*Proof.* Exercise.

**FACT:** Every Boolean algebra is a Heyting algebra.  
*Proof.* Exercise.