Predicate Logic: Soundness and Completeness

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consistency

Γ is a set of WFF’s.

▶ Γ is **consistent** iff Γ \not\vdash \bot.

▶ **FACT.** The following three conditions are equivalent:

1. Γ is consistent.
2. For no WFF ϕ is it the case that both Γ ⊢ ϕ and Γ ⊢ \neg ϕ.
3. There is at least one WFF ϕ such that Γ \not\vdash ϕ.

▶ **ALTERNATIVE FACT.** The following conditions are equivalent:

4. Γ is inconsistent.
5. There is a WFF ϕ such that both Γ ⊢ ϕ and Γ ⊢ \neg ϕ.
6. For every WFF ϕ, it holds that Γ ⊢ ϕ.

▶ **Proof.**

(4) ⇒ (6): Let Γ ⊢ \bot. By the rule “⊥ elimination”, we add one more step in the proof to obtain Γ ⊢ ϕ, which holds for every ϕ.

(6) ⇒ (5): Immediate.

(5) ⇒ (4): By the rule “¬ elimination”, from the derivations Γ ⊢ ϕ and Γ ⊢ \neg ϕ, we get Γ ⊢ \bot.
consistency (continued)

**Theorem.** Let $\Gamma$ be a set of WFF's and $\varphi$ a WFF.

- $\Gamma \cup \{ \varphi \}$ is inconsistent iff $\Gamma \vdash \neg \varphi$.
- $\Gamma \cup \{ \varphi \}$ is consistent iff $\Gamma \not\vdash \neg \varphi$. 
Theorem. Let $\Gamma$ be a set of WFF’s and $\varphi$ a WFF.

If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$. (most common form for “soundness”)

Proof. Left to you. Consult also how each deduction rule is justified [LCS, Section 2.3].

Another form for “soundness” is the following:

Corollary. If $\Gamma$ is satisfiable, then $\Gamma$ is consistent.

Proof. Suppose $\Gamma$ is inconsistent. Then there is a WFF $\varphi$ such that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$. By the previous theorem, both $\Gamma \models \varphi$ and $\Gamma \models \neg \varphi$, which is a contradiction.
One form of “completeness” is the following:

**Theorem.** Let $\Gamma$ be a set of sentences (closed WFF’s).
If $\Gamma$ is consistent, then $\Gamma$ is satisfiable.

**Proof.** By the Model-Existence Lemma (lecture notes, which I hope to post soon).

Another form of “completeness”, which is the most common:

**Corollary.**
If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

**Proof.** Suppose $\Gamma \not\models \varphi$. Then $\Gamma \not\models \neg \varphi$. So that $\Gamma \cup \{\neg \varphi\}$ is consistent. By the theorem above, there is a model $\mathcal{M}$ of $\Gamma \cup \{\neg \varphi\}$. Hence, $\mathcal{M}$ is a model of $\Gamma$ but not of $\varphi$. Hence, $\Gamma \not\models \varphi$. 
soundness and completeness – short form

For all WFF $\varphi$

⊢ $\varphi$ if and only if $\models \varphi$