CS 512, Spring 2015, Handout 13

Predicate Logic:
Prenex Normal Form and Related Topics

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Lemma. For any string of quantifiers $\Phi_{x_1} \equiv \Phi_1 x_1 \Phi_2 x_2 \cdots \Phi_n x_n$ where $\Phi_1, \Phi_2, \ldots, \Phi_n \in \{\forall, \exists\}$, and for any WFF's $\phi$ and $\psi$:

\[
\begin{align*}
\vdash \Phi_{x} \neg \forall y \phi & \leftrightarrow \Phi_{x} \exists y \neg \phi \\
\vdash \Phi_{x} \neg \exists y \phi & \leftrightarrow \Phi_{x} \forall y \neg \phi \\
\vdash \Phi_{x} (\forall y \phi \lor \psi) & \leftrightarrow \Phi_{x} (\forall z (\phi[y:=z] \lor \psi)) \\
\vdash \Phi_{x} (\phi \lor \forall y \psi) & \leftrightarrow \Phi_{x} (\forall z (\phi \lor \psi[y:=z])) \\
\vdash \Phi_{x} (\exists y \phi \lor \psi) & \leftrightarrow \Phi_{x} (\exists z (\phi[y:=z] \lor \psi)) \\
\vdash \Phi_{x} (\phi \lor \exists y \psi) & \leftrightarrow \Phi_{x} (\exists z (\phi \lor \psi[y:=z]))
\end{align*}
\]

where $z$ is a fresh variable occurring nowhere else.

Proof. Similar to proof of Theorem 2.13 in LCS, page 117.
more on quantifier equivalences

**Lemma.** For any string of quantifiers

\[ \overrightarrow{Qx} \triangleq Q_1x_1 Q_2x_2 \cdots Q_nx_n \]

where \( Q_1, Q_2, \ldots, Q_n \in \{\forall, \exists\} \), and for any WFF’s \( \varphi \) and \( \psi \):

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more on quantifier equivalences

**Lemma.** For any string of quantifiers

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6. \( \vdash \overrightarrow{Qx} (\varphi \lor \exists y \psi) \leftrightarrow \overrightarrow{Qx} \exists z (\varphi \lor \psi[y:=z]) \)

where \( z \) is a fresh variable occurring nowhere else.
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where \( z \) is a fresh variable occurring nowhere else.

Proof. Similar to proof of Theorem 2.13 in LCS, page 117.
Theorem. For every WFF $\phi$ there is an equivalent WFF $\psi$ with the same free variables where all quantifiers appear at the beginning. $\psi$ is called the prenex normal form of $\phi$.

Proof. By induction on the structure of $\phi$.

▶ If $\phi$ is atomic, then $\psi \equiv \phi$.

▶ If $\phi$ is $Qx\phi_0$ where $Q \in \{\forall, \exists\}$ and $\psi_0$ is a PNF of $\phi_0$, then $\psi \equiv Qx\psi_0$.

▶ If $\phi$ is $\neg\phi_0$ and $\psi_0$ is a PNF of $\phi_0$, then use the two first cases in the lemma repeatedly, to obtain $\psi$.

▶ If $\phi$ is $\phi_0 \lor \phi_1$, and $\psi_0$ and $\psi_1$ are PNF's of $\phi_0$ and $\phi_1$, then use the four last cases in the lemma repeatedly, to obtain $\psi$. 
**Theorem.** For every WFF $\varphi$ there is an equivalent WFF $\psi$ with the same free variables where all quantifiers appear at the beginning. $\psi$ is called the **prenex normal form** of $\varphi$. 
Theorem. For every WFF \( \varphi \) there is an equivalent WFF \( \psi \) with the same free variables where all quantifiers appear at the beginning. \( \psi \) is called the prenex normal form of \( \varphi \).

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Lemma. A WFF \( \varphi \) of the form
\[
\varphi \equiv \forall x_1 \cdots \forall x_n \exists y \psi
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on vocabulary/signature \( \Sigma \) is equisatisfiable with the WFF
\[
\varphi' \equiv \forall x_1 \cdots \forall x_n \psi \left[ y := f(x_1, \ldots, x_n) \right]
\]
where \( f \) is a fresh \( n \)-ary function symbol not in \( \Sigma \).

Proof. Let \( M \) be a model for \( \Sigma \) and \( M' \equiv (M, f_M) \) a model for \( \Sigma \cup \{ f \} \). If \( M' \models \varphi' \) then \( M \models \varphi \). Hence, if \( \varphi' \) is satisfiable, then so is \( \varphi \).

Conversely, let \( M \models \varphi \). Construct a model \( M' \) for \( \Sigma \cup \{ f \} \) by expanding \( M \) so that for every \( a_1, \ldots, a_n \in A \), the function \( f_M' \) maps \( (a_1, \ldots, a_n) \) to \( b \) where \( M, a_1, \ldots, a_n, b \models \psi \). Hence, \( M' \models \varphi' \). Hence, if \( \varphi \) is satisfiable, then so is \( \varphi' \).
Lemma. A WFF $\varphi$ of the form

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**Proof.**

Let $\mathcal{M}$ be a model for $\Sigma$ and $\mathcal{M}' \triangleq (\mathcal{M}, f^{\mathcal{M}'} )$ a model for $\Sigma \cup \{f\}$. If $\mathcal{M}' \models \varphi'$ then $\mathcal{M} \models \varphi$. Hence, if $\varphi'$ is satisfiable, then so is $\varphi$. 
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Theorem.
If $\phi$ is a first-order sentence over the vocabulary/signature $\Sigma$, then there is a universal first-order sentence $\phi'$ over an expanded vocabulary/signature $\Sigma'$ obtained by adding new function symbols such that $\phi$ and $\phi'$ are equisatisfiable.

Proof.
By repeated use of the lemma.

Remark.
The theorem does NOT claim that $\phi$ and $\phi'$ are equivalent, only that they are equisatisfiable. However, it will be always the case that $\vdash \phi' \rightarrow \phi$, but not always that $\vdash \phi \rightarrow \phi'$.
Theorem. If $\varphi$ is a first-order sentence over the vocabulary/signature $\Sigma$, then there is a **universal** first-order sentence $\varphi'$ over an expanded vocabulary/signature $\Sigma'$ obtained by adding new function symbols such that $\varphi$ and $\varphi'$ are equisatisfiable.
Theorem. If \( \varphi \) is a first-order sentence over the vocabulary/signature \( \Sigma \), then there is a universal first-order sentence \( \varphi' \) over an expanded vocabulary/signature \( \Sigma' \) obtained by adding new function symbols such that \( \varphi \) and \( \varphi' \) are equisatisfiable.

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Exercise:

Let $\varphi(x, y)$ be an atomic WFF with free variables $x$ and $y$, and $f$ a unary function symbol not appearing in $\varphi$. 
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1. Show that the following sentence is valid, i.e., formally provable:

$$\forall x \varphi(x, f(x)) \rightarrow \forall x \exists y \varphi(x, y)$$

*Hint:* You can use any of the available methods, i.e., you can try to find a formal proof or you can try a semantic approach to show $\forall x \varphi(x, f(x)) \models \forall x \exists y \varphi(x, y)$.

2. Show that the following sentence is NOT valid:

$$\forall x \exists y \varphi(x, y) \rightarrow \forall x \varphi(x, f(x))$$

*Hint:* Try a semantic approach, i.e., define an appropriate $\varphi$ and a model where the left-hand side of "$\rightarrow$" is true but the right-hand side of "$\rightarrow$" is false.
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