Counterexamples – in general

(material in this and later slides mostly due to Prof. J-P Katoen of Aachen Univ)

- Reminder: **model checking** $=$ **bug hunting**, bugs are discovered by **counterexamples**, states that refute a given property (desirable or harmful).

- **Counterexamples** are (formally expressed) instances of system behavior that contradict a system’s (formally expressed) specification.
Counterexamples – in general

- **Counterexamples in LTL** are typically finite execution paths:
  - To contradict $(G \varphi)$, we want a finite path ending in a $(\neg \varphi)$-state.
  - To contradict $(F \varphi)$, we want a finite $(\neg \varphi)$-path leading to a $(\neg \varphi)$-cycle.

Methods of LTL model-checkers incorporate forms of **breadth-first search** for generating shortest counterexamples (e.g., see Handout 23).
Counterexamples – in general

- **Counterexamples in LTL** are typically finite execution paths:
  - To contradict $G \varphi$, we want a finite path ending in a $(\neg \varphi)$-state.
  - To contradict $F \varphi$, we want a finite $(\neg \varphi)$-path leading to a $(\neg \varphi)$-cycle.

Methods of LTL model-checkers incorporate forms of **breadth-first search** for generating shortest counterexamples (e.g., see Handout 23).

- **Counterexamples in CTL** are typically finite trees of execution paths:
  - To contradict universal CTL, we want **all** paths in a tree of execution paths.
  - To contradict existential CTL, we want **one** path in a tree of execution paths.

Methods of CTL model-checkers also incorporate some form of **breadth-first search**, combined with more advanced data structures.
Problem statement:

Given a WFF of PCTL of the form $P \leq p(\varphi)$

– for example, in shorthand, $(p \ U \leq 1/2 q)$ or $(X \leq 2/3 p)$ –

...
Problem statement:

Given a WFF of PCTL of the form $P \leq p(\varphi)$ — for example, in shorthand, $(p \ U \leq 1/2 q)$ or $(X \leq 2/3 p)$ — together with a Markov chain $M$ and a state $s$ in $M$, we want to decide whether:

$M, s \not\models P \leq p(\varphi)$ or, more succinctly, $s \not\models P \leq p(\varphi)$

A counterexample $C$ for $P \leq p(\varphi)$ at state $s$ in $M$ is a set of finite paths (or evidence) in $M$ satisfying:

- if $\pi \in C$, then $\pi$ starts at $s$ and $\pi \models \varphi$, and
- $\Pr(C) > p$ where $\Pr(C) \triangleq \sum_{\pi \in C} \Pr(\pi)$,
  i.e., the sum of the probabilities of the paths in $C$, exceeds $p$.

If $\Pr(C) > p$, we conclude that $s \not\models P \leq p(\varphi)$. 
Problem statement:

Given a WFF of PCTL of the form $P_{\leq p} (\varphi)$ – for example, in shorthand, $(p \ U_{\leq 1/2} q)$ or $(X_{\leq 2/3} p)$ – together with a Markov chain $M$ and a state $s$ in $M$, we want to decide whether:

$M, s \not\models P_{\leq p} (\varphi)$ or, more succinctly, $s \not\models P_{\leq p} (\varphi)$

A counterexample $C$ for $P_{\leq p} (\varphi)$ at state $s$ in $M$ is a set of finite paths (or evidences) in $M$ satisfying:

- if $\pi \in C$, then $\pi$ starts at $s$ and $\pi \models \varphi$, and
- $Pr(C) > p$ where $Pr(C) \triangleq \sum_{\pi \in C} Pr(\pi)$, i.e., the sum of the probabilities of the paths in $C$, exceeds $p$.

If $Pr(C) > p$, we conclude that $s \not\models P_{\leq p} (\varphi)$.

In this handout, we limit attention to discrete-time Markov chains – we delay work done on continuous-time Markov chains till next year (!).
A counterexample $C$ for $\mathcal{P}_{\leq p}(\varphi)$ is \textbf{minimal} if $|C| \leq |C'|$ for any counterexample $C'$ for $\mathcal{P}_{\leq p}(\varphi)$.
A counterexample $C$ for $\mathcal{P}_{\leq p}(\varphi)$ is **minimal** if $|C| \leq |C'|$ for any counterexample $C'$ for $\mathcal{P}_{\leq p}(\varphi)$.

A counterexample $C$ for $\mathcal{P}_{\leq p}(\varphi)$ is **smallest** if $C$ is minimal and $\Pr(C) \geq \Pr(C')$ for any minimal counterexample $C'$ for $\mathcal{P}_{\leq p}(\varphi)$. 
Counterexamples – in PCTL (Probabilistic CTL)

- A counterexample $C$ for $P_{\leq p}(\varphi)$ is **minimal** if $|C| \leq |C'|$ for any counterexample $C'$ for $P_{\leq p}(\varphi)$.

- A counterexample $C$ for $P_{\leq p}(\varphi)$ is **smallest** if $C$ is minimal and $\Pr(C) \geq \Pr(C')$ for any minimal counterexample $C'$ for $P_{\leq p}(\varphi)$.

- **Fact**: Counterexamples for non-strict probability bounds (i.e., bounds of the form “$\leq p$”, not “$< p$”) are **finite**.
A counterexample $C$ for $\mathcal{P}_{\leq p}(\varphi)$ is **minimal** if $|C| \leq |C'|$ for any counterexample $C'$ for $\mathcal{P}_{\leq p}(\varphi)$.

A counterexample $C$ for $\mathcal{P}_{\leq p}(\varphi)$ is **smallest** if $C$ is minimal and $\Pr(C) \geq \Pr(C')$ for any minimal counterexample $C'$ for $\mathcal{P}_{\leq p}(\varphi)$.

**Fact:** Counterexamples for non-strict probability bounds (i.e., bounds of the form “$\leq p$”, not “$< p$”) are **finite**.

**Infinite** counterexamples may be needed for WFF’s with strict probability bounds.

For example, an **infinite** counterexample is needed for $s_0 \nRightarrow \mathcal{P}_{< 1}(F a)$, i.e., for $s_0 \nRightarrow (F^{< 1} a)$ in the following Markov chain:

```
\begin{array}{c}
S_0 \\
\quad \downarrow \frac{1}{2} \\
\quad \quad \quad \quad \quad \text{a} \\
\quad \quad \quad \quad \quad \uparrow 1 \\
\quad \quad \quad \quad \quad \downarrow \frac{1}{2} \\
S_1
\end{array}
```
Example showing how to handle “until” WFF’s in PCTL.
Example showing how to handle “until” WFF’s in PCTL

\[
\begin{align*}
\text{Wanted:} & \quad \neg (\varphi \mathrel{U}^{1/2} \psi) \\
\text{counterexamples for } s_0 & \not\models (\varphi \mathrel{U}^{1/2} \psi)
\end{align*}
\]

**Diagram:**

- **States:**
  - Blue states: only prop WFF $\varphi$ holds,
  - Red states: only prop WFF $\psi$ holds,
  - Yellow states: neither $\varphi$ nor $\psi$ hold.

**Graph:**

- Nodes: $s_0$, $s_1$, $s_2$, $t_1$, $t_2$.
- Edges with probabilities:
  - $s_0 \rightarrow s_1$: 0.6
  - $s_0 \rightarrow u$: 0.1
  - $s_1 \rightarrow s_2$: 0.667
  - $s_1 \rightarrow t_1$: 0.333
  - $s_1 \rightarrow s_1$: 0.2
  - $s_2 \rightarrow s_1$: 0.2
  - $s_2 \rightarrow t_2$: 0.3
  - $t_1 \rightarrow t_2$: 0.1
  - $t_2 \rightarrow t_2$: 1

Assaf Kfoury, CS 512, Spring 2015, Handout 24
Example showing how to handle “until” WFF’s in PCTL

Wanted:
counterexamples for $s_0 \not\Vdash (\varphi \mathcal{U}_{1/2} \psi)$

<table>
<thead>
<tr>
<th>evidence</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_1 \triangleq s_0 s_1 t_1$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\pi_2 \triangleq s_0 s_1 s_2 t_1$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\pi_3 \triangleq s_0 s_2 t_1$</td>
<td>0.15</td>
</tr>
<tr>
<td>$\pi_4 \triangleq s_0 s_1 s_2 t_2$</td>
<td>0.12</td>
</tr>
<tr>
<td>$\pi_5 \triangleq s_0 s_2 t_2$</td>
<td>0.09</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

blue states: only prop WFF $\varphi$ holds,
red states: only prop WFF $\psi$ holds,
yellow states: neither $\varphi$ nor $\psi$ hold.
Example showing how to handle “until” WFF’s in PCTL

Wanted: counterexamples for $s_0 \not\models (\varphi \mathsf{U}^{1/2} \psi)$

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<tr>
<td>...</td>
<td>...</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>counterexample</th>
<th>cardinality</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>${ \pi_1, \pi_2, \pi_3, \pi_4, \pi_5 }$</td>
<td>5</td>
<td>0.76</td>
</tr>
<tr>
<td>${ \pi_1, \pi_3, \pi_4, \pi_5 }$</td>
<td>4</td>
<td>0.56</td>
</tr>
<tr>
<td>${ \pi_2, \pi_3, \pi_4, \pi_5 }$</td>
<td>4</td>
<td>0.76</td>
</tr>
<tr>
<td>${ \pi_1, \pi_2, \pi_4 }$</td>
<td>3</td>
<td>0.52</td>
</tr>
<tr>
<td>minimal $\rightarrow$ ${ \pi_1, \pi_2, \pi_4 }$</td>
<td>3</td>
<td>0.55</td>
</tr>
</tbody>
</table>

blue states : only prop WFF $\varphi$ holds,
red states : only prop WFF $\psi$ holds,
yellow states : neither $\varphi$ nor $\psi$ hold.
Example showing how to handle “until” WFF’s in PCTL

Wanted:
counterexamples for \( s_0 \not\models (\varphi \mathrel{U}^{1/2} \psi) \)

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</tr>
<tr>
<td>...</td>
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</tbody>
</table>

blue states: only prop WFF \( \varphi \) holds,
red states: only prop WFF \( \psi \) holds,
yellow states: neither \( \varphi \) nor \( \psi \) hold.

counterexample | cardinality | probability |
<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
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<tr>
<td>{ \pi_1, \pi_2, \pi_3, \pi_4, \pi_5 }</td>
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<td>{ \pi_1, \pi_2, \pi_3 }</td>
<td>3</td>
<td>0.55</td>
</tr>
</tbody>
</table>

smallest
Step 1: Make all $\psi$-states and all $(\neg \varphi \land \neg \psi)$-states absorbing, which requires eliminating some transitions (e.g., the transitions out of $t_1$ and $u$) and making the transition probability $= 1$ on all self-loops.
Adapting a bit more

Step 2: Insert a sink state and redirect all outgoing edges of $\psi$-states to it.
Step 3: Turn the Markov chain into a weighted digraph (directed graph), where:

\[ w(s, s') \triangleq \log \left( \frac{1}{\Pr(s, s')} \right) \]

for every pair of nodes/states \( s \) and \( s' \). The logarithm can be base 10, or base \( e \), or base 2 – it does not matter which base we choose.
A simple derivation

Given a finite path $\pi \triangleq s_0 s_1 s_2 \cdots s_n$:

$$w(\pi) = w(s_0, s_1) + w(s_1, s_2) + \cdots + w(s_{n-1}, s_n)$$

$$= \log \left( \frac{1}{\Pr(s_0, s_1)} \right) + \log \left( \frac{1}{\Pr(s_1, s_2)} \right) + \cdots + \log \left( \frac{1}{\Pr(s_{n-1}, s_n)} \right)$$

$$= \log \left( \frac{1}{\Pr(s_0, s_1) \cdot \Pr(s_1, s_2) \cdots \Pr(s_{n-1}, s_n)} \right)$$

$$= \log \left( \frac{1}{\Pr(\pi)} \right)$$

**Conclusion 1:** For all finite paths $\pi$ and $\pi'$ in the Markov chain, we have:

$$\Pr(\pi) \geq \Pr(\pi')$$ if and only if $$w(\pi) \leq w(\pi')$$

in the Markov chain in the weighted digraph

**Conclusion 2:** Finding a **strongest evidence** in the Markov chain is translated to a **shortest path problem** in the weighted digraph.
Another example: How to handle reachability properties

Wanted: counterexamples for $P_{\leq 0.4}(F \varphi)$, or, in shorthand, $(F_{\leq 0.4} \varphi)$.

![Graph showing transition probabilities and states]

**blue state**: only one $\varphi$-state.
Another example: How to handle reachability properties

Wanted: counterexamples for $P_{\leq 0.4}(F\varphi)$, or, in shorthand, $(F^{\leq 0.4\varphi})$.

Approach 1, based on using the transition (right-stochastic) $9 \times 9$ matrix $A$:

$$
\begin{bmatrix}
s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\
s_0 & 0 & .5 & .25 & 0 & 0 & .25 & 0 & 0 & 0 \\
s_1 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 \\
s_2 & 0 & .5 & 0 & 0 & .5 & 0 & 0 & 0 & 0 \\
s_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
s_4 & 0 & .7 & 0 & .3 & 0 & 0 & 0 & 0 & 0 \\
s_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
s_6 & 0 & 0 & 0 & .5 & 0 & 0 & 0 & .5 & 0 \\
s_7 & 0 & 0 & 0 & 0 & 0 & .25 & .25 & 0 & .5 \\
s_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}
$$

blue state: only one $\varphi$-state.
Another example: How to handle reachability properties

Wanted: counterexamples for \( P_{\leq 0.4} (F \varphi) \), or, in shorthand, \( (F \leq 0.4 \varphi) \).

Approach 1, based on using the transition (right-stochastic) \( 9 \times 9 \) matrix \( A \):

\[
\begin{bmatrix}
 s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\
 s_0 & 0 & .5 & .25 & 0 & 0 & .25 & 0 & 0 & 0 \\
 s_1 & 0 & 0 & .5 & 0 & .5 & 0 & 0 & 0 & 0 \\
 s_2 & 0 & .5 & 0 & 0 & .5 & 0 & 0 & 0 & 0 \\
 s_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 s_4 & 0 & .7 & 0 & .3 & 0 & 0 & 0 & 0 & 0 \\
 s_5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 s_6 & 0 & 0 & 0 & .5 & 0 & 0 & 0 & .5 & 0 \\
 s_7 & 0 & 0 & 0 & 0 & 0 & .25 & .25 & 0 & .5 \\
 s_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

blue state: only one \( \varphi \)-state.

- initial distribution over 9 states is \( d_0 = (1, 0, 0, 0, 0, 0, 0, 0, 0) = [1 0 0 0 0 0 0 0 0] \).
- distribution after at most 1 transition, 2 transitions, and 3 transitions:
  \[
  d_1 = d_0 \cdot A = (0, .5, .25, 0, 0, .25, 0, 0, 0)
  \]
  \[
  d_2 = d_0 \cdot A^2 = (0, .125, .25, .25, .125, 0, .25, 0, 0)
  \]
  \[
  d_3 = d_0 \cdot A^3 = (0, .2125, .0625, .475, .125, 0, 0, .125, 0)
  \]
Another example: How to handle reachability properties

**Wanted:** counterexamples for $\mathcal{P}_{\leq 0.4}(F \varphi)$, or, in shorthand, $(F^{\leq 0.4} \varphi)$.

**Approach 1,** based on using the transition (right-stochastic) $9 \times 9$ matrix $A$:

\[
\begin{bmatrix}
  s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\
  s_0 & 0 & .5 & .25 & 0 & 0 & .25 & 0 & 0 & 0 \\
  s_1 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 \\
  s_2 & 0 & .5 & 0 & 0 & .5 & 0 & 0 & 0 & 0 \\
  s_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  s_4 & 0 & .7 & 0 & .3 & 0 & 0 & 0 & 0 & 0 \\
  s_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  s_6 & 0 & 0 & 0 & .5 & 0 & 0 & 0 & .5 & 0 \\
  s_7 & 0 & 0 & 0 & 0 & 0 & .25 & .25 & 0 & .5 \\
  s_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

**Conclusion:** Starting from $s_0$, state $s_3$ is reached with probability $.475 > .4$ after at most 3 transitions.

- Hence, there is a counterexample $C$ for $s_0 \not\models (F^{\leq 0.4} \varphi)$ consisting of finite paths, each with at most 3 transitions – but we have not determined the members of the counterexample $C$ yet, nor do we know if it is **minimal** or **smallest** (cf. page 8)
Another example: How to handle reachability properties

Wanted: counterexamples for $\mathcal{P}_{\leq 0.4}(\mathbf{F} \varphi)$, or, in shorthand, $(\mathbf{F} \leq 0.4 \varphi)$.

▶ **Approach 2:** Let $S$ be the set of states in the Markov chain, $s_0 \in S$ a single initial state, and $\text{Target} \subseteq S$ a non-empty set of target states.

For every state $s$, we define the probability $p_s$ of reaching the states in $\text{Target}$ from $s$:

$$p_s \triangleq \begin{cases} 
1 & \text{if } s \in \text{Target}, \\
0 & \text{if no state in } \text{Target} \text{ is reachable from } s, \\
\sum_{s' \in S} \Pr(s, s') \cdot p_{s'} & \text{otherwise.}
\end{cases}$$

▶ This defines a system of linear equations over the variables $V \triangleq \{ p_s \mid s \in S \}$ whose unique solution $\sigma : V \to [0, 1]$ assigns to each $p_s$ the probability of reaching $\text{Target}$ from $s$.

▶ Hence, $\mathcal{M} \models \mathcal{P}_{\leq \rho}(\mathbf{F} \text{ target})$ iff $\sigma(p_{s_0}) \leq \rho$, where “target” is an atomic proposition which labels every state in $\text{Target}$.

▶ **Advantage of Approach 2 over Approach 1:** Solving a system of linear equations instead of repeatedly multiplying stochastic matrices.
Another example: How to handle reachability properties

Wanted: counterexamples for $P_{\leq 0.4}(F \varphi)$, or, in shorthand, $(F_{\leq 0.4} \varphi)$.

▶ For the Markov chain $\mathcal{M}$ shown on slide 21, we obtain:

\[
\begin{align*}
p_{s_0} &= 0.5 p_{s_1} + 0.25 p_{s_2} + 0.25 p_{s_5} & p_{s_1} &= 0.5 p_{s_2} + 0.5 p_{s_3} \\
p_{s_2} &= 0.5 p_{s_1} + 0.5 p_{s_4} & p_{s_3} &= 1 \\
p_{s_4} &= 0.7 p_{s_1} + 0.3 p_{s_3} & p_{s_5} &= 1 p_{s_6} \\
p_{s_6} &= 0.5 p_{s_3} + 0.5 p_{s_7} & p_{s_7} &= 0.25 p_{s_5} + 0.25 p_{s_6}
\end{align*}
\]

We can remove all states from $\mathcal{M}$ which do not reach states in Target. In this example, we remove $s_8$, thus also removing equation $p_{s_8} = 0$.

▶ Solving the system of linear equations (using Matlab or Octave), we obtain a solution $\sigma : \{p_{s_0}, p_{s_1}, \ldots, p_{s_7}\} \rightarrow [0, 1]$ such that:

\[
\begin{align*}
\sigma(p_{s_0}) &= 11/12 & \sigma(p_{s_1}) &= \sigma(p_{s_2}) &= \sigma(p_{s_3}) &= \sigma(p_{s_4}) &= 1 \\
\sigma(p_{s_5}) &= \sigma(p_{s_6}) &= 2/3 & \sigma(p_{s_7}) &= 1/3
\end{align*}
\]

▶ Conclusion: Starting from $s_0$, state $s_3$ is reached with probability $\frac{11}{12} > .4$

Hence, there is a counterexample $C$ for $s_0 \not\models (F_{\leq 0.4} \varphi)$, though we do not know the members of $C$ yet!!
Another example: How to handle reachability properties

Wanted: counterexamples for $\mathcal{P}_{\leq 0.4}(\mathcal{F} \varphi)$, or, in shorthand, $(\mathcal{F}^{\leq 0.4} \varphi)$.

- **Approach 3**, most efficient and most direct, repeats the steps carried out to find counterexamples for $s_0 \not\models (\varphi \mathcal{U}^{1/2} \psi)$, from slide 12 to slide 20.

- We obtain, in order of decreasing probabilities:

<table>
<thead>
<tr>
<th>evidence</th>
<th>weight (rounded)</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_1 \triangleq s_0 \ s_1 \ s_3$</td>
<td>1.39</td>
<td>0.25</td>
</tr>
<tr>
<td>$\pi_2 \triangleq s_0 \ s_5 \ s_6 \ s_3$</td>
<td>2.08</td>
<td>0.125</td>
</tr>
<tr>
<td>$\pi_3 \triangleq s_0 \ s_2 \ s_1 \ s_3$</td>
<td>2.77</td>
<td>0.0625</td>
</tr>
<tr>
<td>$\pi_4 \triangleq s_0 \ s_1 \ s_2 \ s_1 \ s_3$</td>
<td>2.77</td>
<td>0.0625</td>
</tr>
<tr>
<td>$\pi_5 \triangleq s_0 \ s_2 \ s_4 \ s_1 \ s_3$</td>
<td>3.13</td>
<td>0.04375</td>
</tr>
<tr>
<td>$\pi_6 \triangleq s_0 \ s_1 \ s_2 \ s_4 \ s_1 \ s_3$</td>
<td>3.13</td>
<td>0.04375</td>
</tr>
<tr>
<td>$\pi_7 \triangleq s_0 \ s_2 \ s_4 \ s_3$</td>
<td>3.28</td>
<td>0.03750</td>
</tr>
<tr>
<td>$\pi_8 \triangleq s_0 \ s_1 \ s_2 \ s_4 \ s_3$</td>
<td>3.28</td>
<td>0.03750</td>
</tr>
<tr>
<td>...</td>
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<td>...</td>
</tr>
</tbody>
</table>

where we take weight $w(s, s') \triangleq -\ln(\Pr(s, s'))$ for all states $s, s' \in S$.

- $\sum_{i \in \{1, 2, 3\}} w(\pi_i) = \sum_{i \in \{1, 2, 4\}} w(\pi_i) = 0.4375 > 0.4$

  (but why not $\{\pi_1, \pi_2, s_0 s_2 s_4\}$ or $\{\pi_1, \pi_2, s_0 s_1 s_2 s_4\}$??)

implies both $\{\pi_1, \pi_2, \pi_3\}$ and $\{\pi_1, \pi_2, \pi_4\}$ are smallest counterexamples.