More on LTL, CTL, and CTL*

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Temporal Logics

3 temporal logics were covered in today’s lecture:

- **LTL**, Linear Temporal Logic
- **CTL**, Computation Tree Logic
- **CTL***, a combination of both LTL and CTL

Choosing the logic depends on the domain.

Examples of CTL on handout 22 p1-12, these diagrams may be more intuitive than the formal definitions. The connectives of CTL are similar to LTL but quantified so they work on trees.

The power of these logics is represented in the Venn diagram:

![Venn Diagram]

we will explore all the non obvious cases of this Venn diagram.

Notation

For LTL recall we write

\[ \mathcal{M} \models \varphi \]

when all states of the model \( \mathcal{M} \) satisfy \( \varphi \).

We write

\[ \mathcal{M}, s \models \varphi \]

when the specific state \( s \) of \( \mathcal{M} \) satisfy \( \varphi \)

Finally, we write

\[ s \models \varphi \]

when the model \( \mathcal{M} \) is clear in context

Notation is the same for CTL but there are subtle differences in meaning that will be elaborated on.
When there is ambiguity about what system the model is evaluated in, we will write
\[ M \models_{LTL} \varphi \text{ or } M \models_{CTL} \varphi \] instead of \( M \models \varphi \).

We can compare formulas of LTL and CTL on the same model \( M \).

If \( \varphi \) is a wff of LTL and \( \psi \) is a wff of CTL,
we can say \( \varphi \equiv \psi \), “\( \varphi \) is equivalent to \( \psi \)”, iff for every model \( M \) and every state \( s \) of \( M \), \( M, s \models_{LTL} \varphi \) iff \( M, s \models_{CTL} \psi \).

### Syntax of CTL

Handout 21p24

Every connective comes with a path quantifier.

### Semantics of CTL

Handout 21p3-8

There is a slightly awkward recursive definition for

\[ M, s \models_{CTL} A[aU\beta] \]

on page 6. Since it is defined in terms of LTL,
\[ M, s \models_{CTL} A[aU\beta] \text{ iff for every path we have } s \models_{LTL} aU\beta \text{ for all paths.} \]

How would we evaluate a statement of the form \( A[aUA[bUc]] \)?

By our recursive definition
\[ M, s \models_{CTL} A[aUA[bUc]] \text{ iff we have } s \models_{LTL} aUA[bUc]. \]

But \( aUA[bUc] \) is not LTL. However it is clear from context what the statement intends.

We have only talked about semantics for model theory, there is also a proof theory that we will not cover in class.

### When \( M \models_{LTL} \varphi \) but not, \( M \models_{CTL} \psi \)

For an atomic proposition, \( a \), we can define the LTL wff
\[ \varphi \overset{\text{def}}{=} F(a \land Xa) \]
and the CTL wff
\[ \psi \overset{\text{def}}{=} AF(a \land AXa) \]
we might expect \( \varphi \equiv \psi \). But consider the model
Clearly, $\mathcal{M}, s_0 \models_{LTL} F(a \land Xa)$
but $\mathcal{M}, s_0 \nvDash_{CTL} AF(a \land AXa)$
because the path $s_0, s_1, s_2, s_2...$ since not all immediate children of $s_0$ have the property $a$.

**Another example of when $\mathcal{M} \models_{LTL} \varphi$ but not when $\mathcal{M} \models_{CTL} \psi$**

Consider the model

Clearly, $\mathcal{M} \models_{LTL} FGa$
and $\mathcal{M} \nvDash_{CTL} AFAGa$
Because the path that stays on $s_0$, may have not $a$ in the future.

**Example of when $\mathcal{M} \models_{CTL} \psi$ but not $\mathcal{M} \models_{LTL} \varphi$**
on Handout 21p24

**Example of when no $\mathcal{M} \models_{CTL} \psi$ and no $\mathcal{M} \models_{LTL} \varphi$, that works in $CTL^*$**
on Handout 21p21
Algebraic Laws for LTL

Some additional laws that were not covered in class are

\[ \alpha U \beta \equiv \alpha \lor (\beta \land X (\alpha U \beta)) \]

\[ \mathbf{G} (\alpha \land \beta) \equiv \mathbf{G} \alpha \land \mathbf{G} \beta \]

\[ \mathbf{F} (\alpha \lor \beta) \equiv \mathbf{F} \alpha \lor \mathbf{F} \beta \]

Algebraic Laws for CTL

\[ \mathbf{A} [\alpha U \beta] \equiv \alpha \lor (\beta \land AXA [\alpha U \beta]) \]

\[ \mathbf{A} \mathbf{F} \alpha \equiv \alpha \lor AXAF \alpha \]

\[ \mathbf{A} \mathbf{G} \alpha \equiv \alpha \land AXAG \alpha \]

\[ \mathbf{E} [\alpha U \beta] \equiv \alpha \lor (\beta \land EXE [\alpha U \beta]) \]

\[ \mathbf{E} \mathbf{F} \alpha \equiv \alpha \lor EXEF \alpha \]

\[ \mathbf{E} \mathbf{G} \alpha \equiv \alpha \land EXEG \alpha \]

\[ \mathbf{A} \mathbf{G} (\alpha \land \beta) \equiv \mathbf{A} \mathbf{G} \alpha \land \mathbf{A} \mathbf{G} \beta \]

\[ \mathbf{E} \mathbf{F} (\alpha \lor \beta) \equiv \mathbf{E} \mathbf{F} \alpha \lor \mathbf{E} \mathbf{F} \beta \]