Problem 3 solution has been taken from Sahil, Problem 5 and 6 solutions have been taken from Ben

1 Problem1:

1.1 Problem 1 a): \( P \rightarrow Q, P \rightarrow \neg Q \vdash \neg P \)

\[
\begin{array}{ll}
1 & P \rightarrow Q & \text{premise} \\
2 & P \rightarrow \neg Q & \text{premise} \\
3 & P & \text{assume} \\
4 & Q & \rightarrow \text{e } 1,3 \\
5 & \neg Q & \rightarrow \text{e } 2,3 \\
6 & \bot & \rightarrow \text{e } 4,5 \\
7 & \neg P & \rightarrow \text{i } 3,6 \\
\end{array}
\]
1.2 Problem 1 b): $P \rightarrow (Q \rightarrow R), P, \neg R \vdash \neg Q$

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P \rightarrow (Q \rightarrow R)$</td>
</tr>
<tr>
<td>2</td>
<td>$P$</td>
</tr>
<tr>
<td>3</td>
<td>$\neg R$</td>
</tr>
<tr>
<td>4</td>
<td>$Q \rightarrow R$</td>
</tr>
<tr>
<td>5</td>
<td>$Q$</td>
</tr>
<tr>
<td>6</td>
<td>$R$</td>
</tr>
<tr>
<td>7</td>
<td>$\bot$</td>
</tr>
<tr>
<td>8</td>
<td>$\neg Q$</td>
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</table>

2 Problem 2:

2.1 a)

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<tr>
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<tr>
<td>3</td>
<td>$\neg \neg P$</td>
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<td>$\neg \neg Q \rightarrow \neg \neg P$</td>
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### 2.2 b)

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</tr>
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<td>( \neg P )</td>
<td>assume</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( P \land Q )</td>
<td>assume</td>
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<td>4</td>
<td>( P )</td>
<td></td>
<td>( \land e_1 ) \ 3</td>
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<td></td>
<td>( \neg e ) \ 2, 4</td>
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<td>( \neg (p \land q) )</td>
<td>( \neg i ) \ 3 – 5</td>
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</tr>
<tr>
<td>7</td>
<td>( \neg Q )</td>
<td>assume</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>( P \land Q )</td>
<td>assume</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>( Q )</td>
<td></td>
<td>( \land e_2 ) \ 8</td>
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<td>( \bot )</td>
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<td>( \neg e ) \ 7, 9</td>
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<td>( \neg (p \land q) )</td>
<td>( \lor e ) \ 1, 2 – 6, 7 – 11</td>
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### 2.3 c)

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<td>premise</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( P )</td>
<td>assume</td>
<td></td>
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<tr>
<td>4</td>
<td>( \bot )</td>
<td></td>
<td>( \neg e ) \ 1, 3</td>
</tr>
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<td>5</td>
<td>( Q )</td>
<td></td>
<td>( \bot e ) \ 4</td>
</tr>
<tr>
<td>6</td>
<td>( Q )</td>
<td>assume</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>( Q )</td>
<td></td>
<td>( \lor e ) \ 2, 3 – 5, 6</td>
</tr>
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</table>
2.4  d)

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<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>$\neg Q \lor R$</td>
<td>premise</td>
</tr>
<tr>
<td>3</td>
<td>$\neg Q$</td>
<td>assume</td>
</tr>
<tr>
<td>4</td>
<td>$P \lor Q$</td>
<td>copy 1</td>
</tr>
<tr>
<td>5</td>
<td>$P$</td>
<td>assume</td>
</tr>
<tr>
<td>6</td>
<td>$P \lor R$</td>
<td>$\lor i$ 5</td>
</tr>
<tr>
<td>7</td>
<td>$Q$</td>
<td>assume</td>
</tr>
<tr>
<td>8</td>
<td>$\bot$</td>
<td>$\neg e$ 3,7</td>
</tr>
<tr>
<td>9</td>
<td>$P \lor R$</td>
<td>$\bot e$ 8</td>
</tr>
<tr>
<td>10</td>
<td>$P \lor R$</td>
<td>$\lor e$ 4, 5, 6, 7, 9</td>
</tr>
<tr>
<td>11</td>
<td>$R$</td>
<td>assume</td>
</tr>
<tr>
<td>12</td>
<td>$P \lor R$</td>
<td>$\lor i$ 11</td>
</tr>
<tr>
<td>13</td>
<td>$P \lor R$</td>
<td>$\lor e$ 2, 3, 10, 11, 12</td>
</tr>
</tbody>
</table>
1 Question 3: taken from Sahil

Consider a parse tree for a given WFF \( \varphi \) with number of literals as \( c \), \( s \) and \( n \). In the tree, all \( s \) atoms lie at the leaf level. All \( n \ \neg \) are immediately following the atoms (like NNF).

In this case, the value of \( n \) becomes irrelevant since it does not affect the height of the parse tree. And we have \( n + s = n \).

The internal nodes, \( c \) logical connectives are all binary operators: \( \land \), \( \lor \) and \( \rightarrow \).

This results into a completely filled binary tree. Such tree with \( X \) leaf nodes has \( (X - 1) \) internal nodes.

\( X \) corresponds to \( s \) and \( (X - 1) \) corresponds to \( c \).

Therefore after substitution, we get \( c = (s - 1) \).

And if we consider \( | \varphi | = l \),

\( c + n + s = l \).
1 Problem 4:

From definition of $\phi^*$ we have
- if $\phi = \alpha \land \beta$ then $\phi^* = \alpha^* \lor \beta^*$
- if $\phi = \alpha \lor \beta$ then $\phi^* = \alpha^* \land \beta^*$
- if $\phi = \neg \alpha$ then $\phi^* = \neg \alpha^*$
- if $\phi = p$ then $\phi^* = \neg p$, that is for $\phi$ with a tree depth 1, $\phi^* \equiv \neg \phi$

Let us assume the tree depth of the formula $\phi$ is $K$ and that for every formula $\psi$ of depth $K' \leq K$ $\psi^* \equiv \neg \psi$ then
- if $\phi = \alpha \land \beta$ then $\phi^* \equiv \neg \alpha \lor \neg \beta \equiv \neg (\alpha \land \beta) \equiv \neg \phi$
- if $\phi = \alpha \lor \beta$ then $\phi^* \equiv \neg \alpha \land \neg \beta \equiv \neg (\alpha \lor \beta) \equiv \neg \phi$
- if $\phi = \neg \alpha$ then $\phi^* \equiv \neg \neg \alpha \equiv \neg \phi$

Hence, $\phi^*$ is tautologically equivalent to $\neg \phi$
5  Page 87, Exercise 1.5.3

5.1 Show that (¬, ∧), (¬, →) and (¬, ⊥) are adequate sets of connectives for propositional logic.

For (¬, ∧):

• Intend to show the following:
  1. \(A \lor B \equiv (\neg A \land \neg B)\)
  2. \(A \rightarrow B \equiv (A \land \neg B)\)

• So for (1), we already know that \(A \lor B \equiv (\neg A \land \neg B)\) from the third truth table in Problem 4.

• For (2), the following truth table will suffice:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(A → B)</th>
<th>¬(A ∧ ¬B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

For (¬, →):

• We only need to show the following:
  1. \(A \lor B \equiv \neg A \rightarrow B\)
  2. \(A \land B \equiv (A \rightarrow \neg B)\)
• For (1), the following truth table will suffice:

<table>
<thead>
<tr>
<th>A B</th>
<th>(( ( A \lor B ) \leftrightarrow ( \neg A \rightarrow B ) ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T T</td>
<td>T T T T T T T T</td>
</tr>
<tr>
<td>T F</td>
<td>T T F T T T F F</td>
</tr>
<tr>
<td>F T</td>
<td>T T T T T T T F</td>
</tr>
<tr>
<td>F F</td>
<td>F F F F F T T T</td>
</tr>
</tbody>
</table>

• For (2):

<table>
<thead>
<tr>
<th>A B</th>
<th>(( ( A \land B ) \leftrightarrow \neg ( A \rightarrow \neg B ) ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T T</td>
<td>T T T T T T T F F</td>
</tr>
<tr>
<td>T F</td>
<td>T F F T T F T T</td>
</tr>
<tr>
<td>F T</td>
<td>T F F F F T T T</td>
</tr>
<tr>
<td>F F</td>
<td>F F F F F T T T</td>
</tr>
</tbody>
</table>

For \((\rightarrow, \bot)\):

• Need to show the following:
  1. \(\neg A \equiv A \rightarrow \bot\)
  2. \(A \land B \equiv ((A \rightarrow (B \rightarrow \bot)) \rightarrow \bot)\)
  3. \(A \lor B \equiv ((A \rightarrow \bot) \rightarrow B)\)

• For (1):

<table>
<thead>
<tr>
<th>A B</th>
<th>(( ( A \rightarrow \bot ) \leftrightarrow \neg A ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T T</td>
<td>T T T T T T</td>
</tr>
<tr>
<td>T F</td>
<td>T F F F F F</td>
</tr>
</tbody>
</table>

• For (2):

<table>
<thead>
<tr>
<th>A B</th>
<th>(( ( A \land B ) \leftrightarrow (( A \rightarrow ( B \rightarrow \bot)) \rightarrow \bot ) ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T T</td>
<td>T T T T T T T T T</td>
</tr>
<tr>
<td>T F</td>
<td>T F T T T F T T T</td>
</tr>
<tr>
<td>F T</td>
<td>T F T T T T T T F</td>
</tr>
<tr>
<td>F F</td>
<td>F F F F F T T T T</td>
</tr>
</tbody>
</table>

• For (3):

<table>
<thead>
<tr>
<th>A B</th>
<th>(( ( A \lor B ) \leftrightarrow (( A \rightarrow \bot ) \rightarrow B ) ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T T</td>
<td>T T T T T T T T</td>
</tr>
<tr>
<td>T F</td>
<td>T T T T T T T T</td>
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<tr>
<td>F T</td>
<td>T T T T T T T T</td>
</tr>
<tr>
<td>F F</td>
<td>F F T T T T T T</td>
</tr>
</tbody>
</table>
5.2 Show that if \( C \subseteq (\neg, \land, \lor, \to, \perp) \) is adequate for propositional logic, then \( \neg \in C \) or \( \perp \in C \).

- Suppose \( C \) contains neither \( \neg \) nor \( \perp \). Then \( C = (\land, \lor, \to) \). Now, consider a simple WFF \( A \land B \), and another \( \neg (A \land B) \):

  \[
  \begin{array}{c|c|c|c}
  A & B & (A \land B) \\
  \hline
  T & T & T \\
  T & \bot & T \\
  \bot & T & \bot \\
  \bot & \bot & \bot \\
  \end{array}
  \]

  \[
  \begin{array}{c|c|c|c}
  A & B & \neg (A \land B) \\
  \hline
  T & T & \bot \\
  T & \bot & \bot \\
  \bot & T & \bot \\
  \bot & \bot & \bot \\
  \end{array}
  \]

- One can see that it would be impossible to express a WFF like \( \neg (A \land B) \) for valuations in which both \( A \) and \( B \) are true using only the connectives \( \land, \lor, \) and \( \to \). The same follows for the formulas \( A \lor B \) and \( A \to B \). That is, using only the connectives \( \land, \lor, \) and \( \to \), we could never construct a WFF which was false when all of it’s propositional atoms were assigned true.

5.3 Is \( (\leftrightarrow, \neg) \) adequate? Prove your answer.

- First, look at the truth table for \( A \leftrightarrow B \), along with \( \neg A \leftrightarrow B \) and \( \neg (A \leftrightarrow B) \):

  \[
  \begin{array}{c|c|c}
  A & B & (A \leftrightarrow B) \\
  \hline
  T & T & T \\
  T & \bot & \bot \\
  \bot & T & \bot \\
  \bot & \bot & T \\
  \end{array}
  \]

  \[
  \begin{array}{c|c|c}
  A & B & \neg (A \leftrightarrow B) \\
  \hline
  T & T & \bot \\
  T & \bot & T \\
  \bot & T & \bot \\
  \bot & \bot & \bot \\
  \end{array}
  \]
From the truth tables above, it is apparent that, using only $\leftrightarrow$ and $\neg$, one can only express WFFs with an even number of valuations in which they are true (the same is true for falsity). But we know from the truth tables from a WFF like $A \leftarrow B$, for example, that there are infinitely many WFFs which have an odd number of valuations for which the WFF is true. Therefore, $\langle \leftrightarrow, \neg \rangle$ is an insufficient set of connectives for propositional logic.

Show that $\langle \#A, \neg \rangle$ is not an adequate set of connectives for propositional logic.

Suppose we were trying to express a WFF with only two propositional atoms, such as $A \land B$. We could try to find an equivalence in an expression like $\langle \#A, \neg A, B \rangle$, but the truth table for that expression includes only an even number of valuations for which it is true (and, conversely, false). And we know that the truth table for $A \land B$ contains only one valuation for which it is true, namely, when both $A$ and $B$ are true. It follows that $A \land B$ is inexpressible using only the set $\langle \#, \neg \rangle$ of connectives, and therefore such a set is insufficient for propositional logic.