CS 512, Spring 2016: Assignment 10
Probabilistic Model Checking

Out: 15 April 2016
Due: Friday, 22 April 2016, by 11:59 pm

• Turn in your solutions electronically, as a single PDF file, by placing it in the subfolder Assignment 10 of the Dropbox folder CS512-Assignments-2016.

• As we draw closer to the end of the semester, in order to encourage timely and readable submissions, we will need to impose two penalties:
  – Late submissions will be penalized 5 points for every day beyond the deadline.
  – Hand-written and scanned submissions will be penalized 10 points.

• The solution scribe and grader for Assignment 10 is: Ben Getchell (bengetch@bu.edu).

Problem 1  Do the last problem (Problem 4) of page 16 in Handout 22. To make the exercise more concrete, show that Randomized DPP is deadlock-free with probability greater than $\frac{9}{10}$.

Hint: You do not need to do any of the three preceding problems on page 16 of Handout 22. It suffices to read and understand the statement of Problem 2 on page 16.

Problem 2  Consider the discrete-time Markov chain $A$ in Figure 1. Using the notation from lecture:

$A \triangleq (S, P, \text{init}, AP, L)$ where

- $S \triangleq \{s_0, s_1, \ldots, s_7\}$ (set of states),
- $P : S \times S \rightarrow [0, 1]$ (probabilistic transition function),
- $\text{init} \triangleq (1, 0, 0, \ldots, 0)$ (initial state distribution, a vector with 8 entries),
- $AP \triangleq \{a, b\}$ (atomic propositions),
- $L : S \rightarrow 2^{AP}$ (labelling function).

As explained in lecture, the transition function $P$ can be conveniently written as a (here $8 \times 8$) right-stochastic (also called row-stochastic) matrix. We consider three WFF’s in the syntax of LTL:

$\varphi_1 \triangleq F (a \land b), \quad \varphi_2 \triangleq X F a, \quad \text{and} \quad \varphi_3 \triangleq G (\varphi_1 \lor \varphi_2)$.

before embedding them in PCTL, as follows:

$\varphi'_1 \triangleq F^{\geq 1/2} (a \land b), \quad \varphi'_2 \triangleq X^{\geq 1/2} F^{=1} a, \quad \text{and} \quad \varphi'_3 \triangleq G^p (\varphi'_1 \lor \varphi'_2)$.

where $p$ is a probability to be yet determined (in part (c) below). There are three parts in this problem:
(a) Compute the three probabilities $p_1$, $p_2$, and $p_3$, defined by:

$$p_1 \triangleq \Pr(A, s_0 \models \varphi_1), \quad p_2 \triangleq \Pr(A, s_0 \models \varphi_2), \quad \text{and} \quad p_3 \triangleq \Pr(A, s_0 \models (\varphi_1 \lor \varphi_2)).$$

(b) Give a rigorous argument whether the two following satisfactions in PCTL hold or not:

$A, \text{init} \models \varphi'_1$ and $A, \text{init} \models \varphi'_2$.

(c) Determine the largest possible probability $p$ such that: $A, \text{init} \models \varphi'_3$.

\[\text{Figure 1: Graphical representation of the discrete-time Markov chain } A \text{ in Problem 2.}\]

Propositional atom $a$ is only true in states $s_4$ and $s_6$, and $b$ is only true in state $s_6$ (the green states).

\[\text{Problem 3} \quad \text{Consider the discrete-time Markov chain } M \text{ on pages 21-24 of Handout 24 and assume:}\]

- no state in $S = \{s_0, s_1, \ldots, s_8\}$, except for state $s_3$, satisfies $\varphi$ (just as in Handout 24), and
- no state in $S = \{s_0, s_1, \ldots, s_8\}$, except for state $s_1$ and $s_3$, satisfies $\psi$,

where $\varphi$ and $\psi$ are propositional WFF's denoting some (unspecified) conditions. Suppose the good functioning of the system modeled by $M$ requires any execution path $\pi \triangleq t_0 \cdot t_1 \cdot t_i \cdot t_{i+1} \cdots$ starting at $t_0 = s_0$ must satisfy one of two properties, ($\dagger$) or ($\ddagger$):

($\dagger$) with probability $\leq 0.4$, path $\pi$ eventually reaches a state $t_i$ satisfying $\varphi$ (just as in Handout 24), or
($\ddagger$) with probability $> 0.8$, path $\pi$ is such that for every state $t_i$, if the immediately succeeding state $t_{i+1}$ satisfies $\psi$ with probability $= 1$, then so does $t_i$ satisfy $\psi$.

Hence, the good functioning of the system requires that $M$ satisfies the following WFF of PCTL:

$$\Phi \triangleq F^{<0.4} \varphi \lor G^{>0.8} (X^{-1} \psi \rightarrow \psi).$$

Your task in this problem is to find a smallest (or, to use a better word, strongest) counterexample $C$ for $\Phi$, which means $M$ must be a finite set of finite paths such that:

(a) $C$ is minimal, i.e., there is no counterexample $C'$ for $\Phi$ with fewer members than $C$, and
(b) $C$ is strongest among all minimal counterexamples for $\Phi$, i.e., the probability $\Pr(C) \triangleq \sum_{\rho \in C} \Pr(\rho)$ is the largest possible.

Use the analysis in Handout 24 as a guide.

Hint 1: You will find it useful to use the fact that $P_{>p}(G \psi)$ or, in shorthand, $(G^{>p} \psi)$ is equivalent to $P_{<1-p}(F \neg \psi)$ or, in shorthand, $(F^{<1-p} \neg \psi)$, for any propositional WFF $\psi$ and any probability $p$. If you use this fact, you will need to argue for it, as a preliminary lemma.
Hint 2: Another fact you may find useful to use is that $(X^{<1} \vartheta)$ is equivalent to $\neg(X^{=1} \vartheta)$. Again here, if you use this fact, you will need to argue for it as a preliminary lemma.

Hint 3: Consider the negation of the sub-WFF in the scope of $G^{>0.8}$, namely, $\neg((X^{=1} \psi) \rightarrow \psi)$. Give a precise argument that there is only one state $s_4$ in $\mathcal{M}$ which satisfies it. Conclude that, for the good functioning of $\mathcal{M}$, the proportion of all execution paths that eventually reach $s_4$ must be at most 0.2.