Problem 1  This problem is based on Sicun Gao’s lecture on Thursday, April 7, 2016. If you followed about half of the lecture, you should be able to answer most of the questions below, if not all of them. The slides for Gao’s lecture are posted on the course webpage:


For full credit, answer any three of the seven questions below \{(a), (b), \ldots, (g)\}.

(a) Let \( f(x) \) be a continuous function in \( x \). We write \( \min(f(x)) \) for the minimum value of \( f \) in its domain. Prove that \( \min(f(x)) < 0 \) if and only if \( \exists x. f(x) < 0 \) is true.

(b) Prove that the first-order theory over real numbers with functions \{\( +, \times, \sin \)\} is undecidable.

For parts \{(c), (d), (e), (f), (g)\} we consider the formula:

\[ \varphi \triangleq (\sin(x) = y) \land (y = x) \land (x \times y \neq 0) \]

where we restrict \( x \) and \( y \) to be in the interval of real numbers between \( 0 \) and \( 1 \), i.e., \( x \in [0, 1] \) and \( y \in [0, 1] \).

You can use the web-interface of dReal (http://dreal.github.io/try/) (or other tools you may want to download and try from the Web) to get the answer.

(c) Is \( \varphi \) satisfiable or not? Why?

(d) Write down the \( \delta \)-weakening of the formula \( \varphi \).

(e) Let \( \delta = 0.001 \). Is the \( \delta \)-weakening of \( \varphi \) satisfiable or not?

(f) Let \( \delta = 0.001 \), but change the domain of \( x \) and \( y \) to \([0.3, 1]\) (instead of \([0, 1]\)). Is the formula \( \varphi \) now satisfiable?

(g) Give a \( \delta \) with two decimal places, such that \( \delta < 1 \) and the formula \( \varphi \) is \( \delta \)-satisfiable over \( x \) and \( y \) ranging in the interval \([0.3, 1]\).
Problem 2  Review pages 13, 14, and 15, in Handout 22. Consider, in particular, the WFF’s of CTL on page 15 as a guide for this problem. In addition to the propositional atoms $e_i$ and $f_i$ (on page 15), you are allowed to introduce one additional propositional atom:

$$w_i \triangleq \text{philosopher}_i \text{ is waiting (to eat), i.e., thinking},$$

but you are not allowed to use propositional atoms other than \{ $e_i$, $f_i$, $w_i$ \}. At every time tick, each of the 4 atoms in \{ $e_i$, $f_i$, $w_i$ \} is true or false. Write CTL WFF’s to express each of the following properties and justify carefully in a couple of lines (especially if your WFF is very complicated!):

(a) It is never the case that all philosophers are waiting (to eat).
(b) Every philosopher gets infinitely often a turn to eat.
(c) No philosopher will starve to death.
(d) If a philosopher is able to eat, then his neighbor to the right will eventually eat.
(e) Always eventually some philosopher is able to eat.
(f) Always eventually every philosopher is able to eat.
(g) It is not possible for neighboring philosophers to eat at the same time.
(h) There is always a philosopher who is eating.

Hint: Some of these properties may be formulated (correctly) in more than one way, using the available propositional atoms.
Problem 3  This is an exercise in formal modeling a problem of concurrency – the Dining Philosophers Problem (DPP) presented in Handout 22, from page 13 and on.

(a) Define the transition system $\mathcal{M}$ (preferably in the form of a diagram) corresponding to DPP, according to the constraints listed on page 13 of Handout 22 but with the following conventions and modifications:

- Assume that every philosopher is required to pick both forks up simultaneously, and again to put them both down simultaneously (when he finishes eating). Also assume that picking the two forks up is instantaneous (or, if you will, it takes a very small negligible amount of time), and again putting them down is instantaneous.
- A state in $\mathcal{M}$ corresponds to a truth assignment to the 5-tuple $(e_1, e_2, e_3, e_4, e_5)$. For example, if $\mathcal{M}$ is in state $(t, f, t, f, f)$, this means that philosophers 1 and 3 are eating and that philosophers 2, 4, and 5 are thinking.
- There is a single start state, call it $s_0$, corresponding to the 5-tuple $(f, f, f, f, f)$, when all the philosophers are thinking (or waiting to eat).
- Limit the state space of $\mathcal{M}$ to all the states that are accessible from $s_0$. In particular, exclude all states that correspond to three or more philosophers eating at the same time; these are not possible and therefore inaccessible. But note there are states with only two philosophers eating which are not possible either, e.g., when the two philosophers are neighbors; these states are inaccessible and should be excluded too.
- Assume that the fixed eating-time interval for all philosophers (see point 4 on page 13 of Handout 22) is exactly one time unit. This implies, in particular, there are no self-loops in the diagram of $\mathcal{M}$ at states other than the start state $s_0$.
- Assume that all 5 philosophers make their moves in synchrony. In particular, moving from one state to an adjacent state in $\mathcal{M}$ means that one time unit has elapsed for all 5 philosophers.
- From the start state $s_0$, there are exactly another 10 states accessible from $s_0$. Denote these 10 states $s_1, s_2, \ldots, s_{10}$. Clearly indicate, at each state in the diagram, which atoms in $\{e_1, w_1, e_2, w_2, \ldots, e_5, w_5\}$ are true and which are false.[1]

(b) For the three first CTL WFF’s on page 15 of Handout 22, decide which are satisfied by model $\mathcal{M}$ and which are not satisfied by $\mathcal{M}$, keeping in mind that you need to limit yourself to execution paths that start with $s_0$ (the only start state). If the WFF is not satisfied, define a path in $\mathcal{M}$ that falsifies it.

(c) Go a little deeper by considering how you need to modify $\mathcal{M}$ to account for the presence of the atoms \{f_1, f_2, \ldots, f_5\}, with the convention that for every $1 \leq i \leq 5$:

- $f_i = f$ at time 0.
- $f_i = t$ at time $t + 1$ iff “$e_i = t$ at time $t$” and “$e_i = f$ at time $t + 1$”, i.e., philosopher $i$ just finished eating at the preceding time unit.
- $f_i = f$ at time $t + 1$ iff “$e_i = f$ at both times $t$ and $t + 1$”, i.e., philosopher $i$ finished eating more than one time unit ago or has not eaten at all.

Assume we start counting at time $= 0$ with discrete time units, and each elapsed time-unit causes $\mathcal{M}$ to move from one state to a neighboring state.

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[1] Ignore the atoms \{f_1, f_2, \ldots, f_5\} on page 15 of Handout 22, because $\mathcal{M}$ as just defined cannot distinguish differences in times elapsed since a philosopher finished eating, i.e., $\mathcal{M}$ does not distinguish between “just finished eating” and “finished two time units ago”. We need a transition system more refined than $\mathcal{M}$ to incorporate \{f_1, f_2, \ldots, f_5\} in its behavior.