from informal/common reasoning to formal reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  **THEN** John is late for the meeting
from informal/common reasoning to formal reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  **THEN** John is late for the meeting

- John is **NOT** late for the meeting
from informal/common reasoning to formal reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  **THEN** John is late for the meeting
- John is **NOT** late for the meeting
- the train did arrive late
from informal/common reasoning to formal reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  **THEN** John is late for the meeting
- John is **NOT** late for the meeting
- the train did arrive late
- **THEREFORE** there were taxis
from informal/common reasoning to formal reasoning:

- IF the train arrives late AND there are NO taxis
  THEN John is late for the meeting
- John is NOT late for the meeting
- the train did arrive late
- THEREFORE there were taxis

again symbolically:
from informal/common reasoning to formal reasoning:

▶ IF the train arrives late AND there are NO taxis
  THEN John is late for the meeting
▶ John is NOT late for the meeting
▶ the train did arrive late
▶ THEREFORE there were taxis

again symbolically:

▶ IF \( P \) AND \( \text{NOT} \ Q \) THEN \( R \)
from informal/common reasoning to formal reasoning:

- IF the train arrives late AND there are NO taxis
  THEN John is late for the meeting
- John is NOT late for the meeting
- the train did arrive late
- THEREFORE there were taxis

again symbolically:

- \(( P \land \neg Q ) \rightarrow R\)
from informal/common reasoning to formal reasoning:

- IF the train arrives late AND there are NO taxis THEN John is late for the meeting
- John is NOT late for the meeting
- the train did arrive late
- THEREFORE there were taxis

again symbolically:

- \(( P \land \neg Q ) \rightarrow R\)
- \(\neg R\)
from informal/common reasoning to formal reasoning:

- IF the train arrives late AND there are NO taxis
  THEN John is late for the meeting
- John is NOT late for the meeting
- the train did arrive late
- THEREFORE there were taxis

again symbolically:

- \(( P \land \neg Q ) \rightarrow R\)
- \(\neg R\)
- \(P\)
from **informal/common** reasoning to **formal** reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  **THEN** John is late for the meeting
- John is **NOT** late for the meeting
- the train did arrive late
- **THEREFORE** there were taxis

again symbolically:

- \(( P \land \neg Q ) \rightarrow R\)
- \(\neg R\)
- \(P\)
- **THEREFORE** \(Q\)
from informal/common reasoning to formal reasoning:

▶ IF the train arrives late AND there are NO taxis
   THEN John is late for the meeting

▶ John is NOT late for the meeting

▶ the train did arrive late

▶ THEREFORE there were taxis

again symbolically:

▶ \(( P \land \neg Q ) \rightarrow R\)

▶ \neg R

▶ P

▶ \vdash Q
from informal/common reasoning to formal reasoning:

- IF the train arrives late AND there are NO taxis
  THEN John is late for the meeting
- John is NOT late for the meeting
- the train did arrive late
- THEREFORE there were taxis

again symbolically:

- \(( P \land \neg Q ) \rightarrow R\)
- \(\neg R\)
- \(P\)
- \(\vdash Q\)

more succinctly:

\[ P \land \neg Q \rightarrow R, \quad \neg R, \quad P \quad \vdash \quad Q \]
a sequent (also called a judgment) is an expression of the form:

\[ \varphi_1, \ldots, \varphi_n \vdash \psi \]

where:

1. \( \varphi_1, \ldots, \varphi_n, \psi \) are well-formed formulas (also called wff’s)

2. the symbol “\( \vdash \)” is pronounced turnstile

3. the wff’s \( \varphi_1, \ldots, \varphi_n \) to the left of “\( \vdash \)” are called the premises (also called antecedents or hypotheses)

4. the wff \( \psi \) to the right of “\( \vdash \)” is called the conclusion (also called succedent)
a sequent is said to be **valid** (also **deducible** or **derivable**) if there is a **formal proof** for it.

a **formal proof** (also called **deduction** or **derivation**) is a sequence of wff’s which starts with the **premises** of the sequent and finishes with the **conclusion** of the sequent:

\[
\varphi_1 \quad \text{premise} \\
\varphi_2 \quad \text{premise} \\
\vdots \\
\varphi_n \quad \text{premise} \\
\vdots \\
\psi \quad \text{conclusion}
\]

where every wff in the deduction is obtained from the wff’s preceding it using a **proof rule**.
Examples of Proof Rules

\[ \frac{\varphi}{\varphi \land \psi} \land i \]
Examples of Proof Rules

\[
\begin{align*}
\phi & \quad \psi \\
\hline
& \phi \land \psi \quad \land i \\
& \phi \land \psi \quad \land e_1
\end{align*}
\]
Examples of Proof Rules

\[ \frac{\varphi \quad \psi}{\varphi \land \psi} \quad \land_i \]

\[ \frac{\varphi \land \psi}{\varphi} \quad \land_{e_1} \]

\[ \frac{\varphi \land \psi}{\psi} \quad \land_{e_2} \]
Examples of Proof Rules

\[
\begin{align*}
\varphi & \quad \psi \\ \hline 
\varphi \land \psi & \quad \land i
\end{align*}
\]

\[
\begin{align*}
\varphi \land \psi \\ \hline 
\varphi & \quad \land e_1
\end{align*}
\]

\[
\begin{align*}
\varphi \land \psi \\ \hline 
\psi & \quad \land e_2
\end{align*}
\]

\[
\begin{align*}
\varphi \\ \hline 
\neg \neg \varphi & \quad \neg \neg i
\end{align*}
\]
Examples of Proof Rules

\[
\begin{array}{c}
\varphi  \\
\psi  \\
\hline
\varphi \land \psi \\
\end{array}
\quad \land i
\]

\[
\begin{array}{c}
\varphi \land \psi  \\
\hline
\varphi  \\
\end{array}
\quad \land e_1
\]

\[
\begin{array}{c}
\varphi \land \psi  \\
\hline
\psi  \\
\end{array}
\quad \land e_2
\]

\[
\begin{array}{c}
\varphi  \\
\hline
\neg \neg \varphi  \\
\end{array}
\quad \neg i
\]

\[
\begin{array}{c}
\neg \neg \varphi  \\
\hline
\varphi  \\
\end{array}
\quad \neg \neg e \quad \text{(cannot be used in intuitionistic logic)}
\]
Examples of Proof Rules

\[ \phi \quad \phi \rightarrow \psi \quad \rightarrow e \quad (\text{or MP for Modus Ponens}) \]
Examples of Proof Rules

1. \( \varphi \vdash \varphi \rightarrow \psi \rightarrow_e \psi \) (or MP for Modus Ponens)
2. \( \varphi \rightarrow \psi \vdash \neg \psi \rightarrow \neg \varphi \) (MT for Modus Tollens)
Examples of Proof Rules

\[ \varphi \quad \varphi \rightarrow \psi \quad \rightarrow \text{e} \quad (\text{or MP for Modus Ponens}) \]

\[ \varphi \rightarrow \psi \quad \neg \psi \quad \rightarrow \neg \varphi \quad \text{MT} \quad (\text{for Modus Tollens}) \]

\[ \varphi \quad \vdots \quad \psi \quad \rightarrow \text{i} \]
Examples of Proof Rules

- $\varphi \quad \varphi \rightarrow \psi \quad \rightarrow e \quad$ (or MP for Modus Ponens)

- $\varphi \rightarrow \psi \quad \neg \psi \quad \neg \varphi \quad$ MT \quad (for Modus Tollens)

- $\varphi \quad \vdots \quad \psi \quad \varphi \rightarrow \psi \quad \rightarrow i$

open a box when you introduce an assumption (wff $\varphi$ in rule $\rightarrow i$)
Examples of Proof Rules

\[ \varphi \rightarrow \psi \]
\[ \Downarrow \]
\[ \varphi \rightarrow \psi \quad \neg \psi \]
\[ \Downarrow \]
\[ \neg \varphi \quad \text{MT} \quad (\text{for Modus Tollens}) \]

open a box when you introduce an assumption (wff \( \varphi \) in rule \( \rightarrow i \))

close the box when you discharge the assumption
Examples of Proof Rules

- \( \vdash \varphi \quad \varphi \rightarrow \psi \quad \rightarrow \psi \quad \text{(or MP for Modus Ponens)} \)

- \( \varphi \rightarrow \psi \quad \neg \psi \quad \neg \varphi \quad \text{MT} \quad \text{(for Modus Tollens)} \)

- \( \vdash \varphi \quad \vdash \psi \quad \vdash \text{...} \quad \varphi \rightarrow \psi \quad \rightarrow \text{i} \)

open a box when you introduce an assumption (wff \( \varphi \) in rule \( \rightarrow \text{i} \))

close the box when you discharge the assumption

you must close every box and discharge every assumption in order to complete a formal proof
Proof Rules Associated with Only One “¬” and with “⊥”

So far, we have an elimination rule and an introduction rule for double negation “¬¬”, namely ¬¬e and ¬¬i, but not for single negation “¬”. We now compensate for this lack:

\[
\frac{\varphi}{\bot} \quad \text{(or LNC for Law of Non-Contradiction)}
\]

where “⊥” (a single symbol) stands for “contradiction”
Proof Rules Associated with Only One “\(\neg\)” and with “\(\bot\)”

So far, we have an elimination rule and an introduction rule for double negation “\(\neg\neg\)” , namely \(\neg\neg e\) and \(\neg\neg i\), but not for single negation “\(\neg\)” . We now compensate for this lack:

\[
\begin{array}{c}
\varphi \\
\hline
\neg \varphi \\
\bot \\
\end{array}
\]

\(\neg e\) ( or LNC for Law of Non-Contradiction)

where “\(\bot\)” (a single symbol) stands for “contradiction”
Proof Rules Associated with Only One “\(\neg\)” and with “\(\bot\)”

So far, we have an **elimination** rule and an **introduction** rule for double negation “\(\neg\neg\)”, namely \(\neg\neg e\) and \(\neg\neg i\), but not for single negation “\(\neg\)”. We now compensate for this lack:

\[
\frac{\varphi}{\bot} \quad \neg e \quad \text{(or LNC for Law of Non-Contradiction)}
\]

where “\(\bot\)” (a single symbol) stands for “contradiction”

\[
\frac{\varphi}{\bot} \quad \neg i
\]

\[
\frac{\bot}{\bot e} \quad \text{ (“if you can prove } \bot \text{, you can prove every WFF”)}
\]
Two Derived Proof Rules

The two following rules are derived rules –
the first from rules →i, ¬i, →e, and ¬¬e (see [LCS, pp 24-25]);
the second from rules ∨i, ¬i, ¬e, and ¬¬e (see [LCS, pp 25-26]):

\[
\begin{array}{c}
\neg \varphi \\
\vdots \\
\bot \\
\hline
\varphi
\end{array}
\]

PBC (for Proof by Contradiction)
Two Derived Proof Rules

The two following rules are derived rules – the first from rules $\to i$, $\neg i$, $\to e$, and $\neg\neg e$ (see [LCS, pp 24-25]); the second from rules $\lor i$, $\neg i$, $\neg e$, and $\neg\neg e$ (see [LCS, pp 25-26]):

\[
\begin{array}{c}
\neg \varphi \\
\vdots \\
\bot \\
\varphi
\end{array} & \text{PBC (for Proof by Contradiction)}
\]

\[
\begin{array}{c}
\varphi \lor \neg \varphi
\end{array} & \text{LEM (for Law of Excluded Middle)}
\]

Because $\neg\neg e$ is rejected in intuitionistic logic, so are PBC and LEM
Two Derived Proof Rules

The two following rules are derived rules –
the first from rules →i, ¬i, →e, and ¬¬e (see [LCS, pp 24-25]);
the second from rules ∨i, ¬i, ¬e, and ¬¬e (see [LCS, pp 25-26]):

\[
\begin{array}{c}
\neg \varphi \\
\vdots \\
\bot \\
\hline
\varphi \\
\end{array}
\]

\[\text{PBC} \quad \text{(for Proof by Contradiction)}\]

\[
\varphi \lor \neg \varphi
\]

\[\text{LEM} \quad \text{(for Law of Excluded Middle)}\]

Because ¬¬e is rejected in intuitionistic logic, so are PBC and LEM

(a summary of all proof rules and some derived rules in [LCS, p. 27])
Examples of Natural Deductions

formal proof of the sequent $P \vdash Q \rightarrow (P \land Q)$
Examples of Natural Deductions

formal proof of the sequent \( P \vdash Q \rightarrow (P \land Q) \)

\[
\begin{array}{c}
1 & P \\
2 & Q \\
3 & P \land Q & \landi 1, 2 \\
4 & Q \rightarrow (P \land Q) & \rightarrowi \\
\end{array}
\]
Examples of Natural Deductions

formal proof of the sequent \( P \rightarrow (Q \rightarrow R) \vdash P \land Q \rightarrow R \)
Examples of Natural Deductions

formal proof of the sequent \( P \rightarrow (Q \rightarrow R) \vdash P \land Q \rightarrow R \)

1. \( P \rightarrow (Q \rightarrow R) \)

2. \( P \land Q \)

3. \( P \)

4. \( Q \rightarrow R \)

5. \( Q \)

6. \( R \)

7. \( P \land Q \rightarrow R \)
Examples of Natural Deductions

formal proof of the sequent \( P \land Q \rightarrow R \vdash P \rightarrow (Q \rightarrow R) \)
Examples of Natural Deductions

formal proof of the sequent $P \land Q \rightarrow R \vdash P \rightarrow (Q \rightarrow R)$

1. $P \land Q \rightarrow R$

2. $P$

3. $Q$

4. $P \land Q \quad \land i \ 2, \ 3$

5. $R \quad \rightarrow e \ 1, \ 4$

6. $Q \rightarrow R \quad \rightarrow i$

7. $P \rightarrow (Q \rightarrow R) \quad \rightarrow i$
Examples of Natural Deductions

formal proof of the sequent \[ P \rightarrow (Q \rightarrow R) \vdash (P \rightarrow Q) \rightarrow (P \rightarrow R) \]
Examples of Natural Deductions

formal proof of the sequent \[ P \rightarrow (Q \rightarrow R) \vdash (P \rightarrow Q) \rightarrow (P \rightarrow R) \]

1. \[ P \rightarrow (Q \rightarrow R) \]

2. \[ P \rightarrow Q \]

3. \[ P \]

4. \[ Q \]

5. \[ Q \rightarrow R \]

6. \[ R \]

7. \[ P \rightarrow R \]

8. \[ (P \rightarrow Q) \rightarrow (P \rightarrow R) \]
Formal Proof of the Initial Sequent:

1. $P \land \neg Q \rightarrow R$  \hspace{1cm} \text{premise}
2. $\neg R$  \hspace{1cm} \text{premise}
3. $P$  \hspace{1cm} \text{premise}
4. $\neg Q$  \hspace{1cm} \text{assume}
5. $P \land \neg Q$  \hspace{1cm} $\land i$ 3, 4
6. $R$  \hspace{1cm} $\rightarrow e$ 1, 5
7. $\bot$  \hspace{1cm} $\neg e$ 6, 2
8. $\neg \neg Q$  \hspace{1cm} $\neg i$
9. $Q$  \hspace{1cm} $\neg \neg e$ 8