Let \( \varphi \triangleq \exists y \left[ P(y) \rightarrow \forall x P(x) \right] \)

\( \varphi \) is a first-order sentence over the vocabulary \( \Sigma = \{P\} \).

Is \( \varphi \) semantically valid (true in every model) or, equivalently, formally provable?

Yes, it is, no matter the interpretation of the predicate symbol \( P \).

So why not consider instead the formula \( \psi \triangleq \forall P \varphi \)?

\( \psi \) is no longer first-order . . . .
from first-order to second-order logic

Given a vocabulary $\Sigma = \mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ as before –

$\mathcal{P}$ is a collection of predicate symbols,
$\mathcal{F}$ a collection of function symbols,
$\mathcal{C}$ a collection of constant symbols –

we go from the syntax and formation rules of first-order logic to second-order logic by adding:

- **predicate variables**: $X_1, X_2, \ldots$ each with a fixed arity $n \geq 1$.
- **function variables**: $F_1, F_2, \ldots$ each with a fixed arity $n \geq 1$.

The definition of a model $\mathcal{M}$ proceeds as in Handout 10, except that now an environment (or look-up table) $\ell$ must assign a meaning to **predicate variables** and **function variables**, in addition to **individual variables**.
from first-order to second-order logic (continued)

The only new features in the definition of satisfaction deal with the second-order quantifiers – see Handout 10:

- let $X$ be a $n$-ary predicate variable, for some $n \geq 1$,

$$\mathcal{M}, \ell \models \forall X \varphi \iff \mathcal{M}, \ell[X \mapsto R] \models \varphi \text{ for every } R \subseteq A \times \cdots \times A$$

- let $F$ be a $n$-ary function variable, for some $n \geq 1$,

$$\mathcal{M}, \ell \models \forall F \varphi \iff \mathcal{M}, \ell[F \mapsto f] \models \varphi \text{ for every } f : A \times \cdots \times A \to A$$
Let $\varphi$ be a second-order WFF. Similar to 1st order logic, we say:

- **WFF $\varphi$ is **satisfiable** iff**
  there is some $M$ and some $\ell$ such that $M, \ell \models \varphi$

- **WFF $\varphi$ is **semantically valid** iff**
  for every $M$ and every $\ell$ it is the case that $M, \ell \models \varphi$

Let $\Gamma$ be a set of second-order WFF’s:

- **$\Gamma$ is **satisfiable** iff**
  there is some $M$ and some $\ell$ such that $M, \ell \models \varphi$ for every $\varphi \in \Gamma$

- **semantic entailment**: $\Gamma \models \psi$ iff for every $M$ and every $\ell$, it holds that $M, \ell \models \Gamma$ implies $M, \ell \models \psi$
soundness and completeness for second-order logic

- There are several deductive systems for second-order logic, but none can be complete w.r.t. second-order semantics.

- At a minimum, each of these deductive systems is sound, i.e., any second-order WFF which is formally derivable is semantically valid.
“A well-ordering is an ordering $\leq$ such that every non-empty set has a least element w.r.t. $\leq$": 

$$\varphi \triangleq \forall X \left[ \exists y X(y) \to \exists v \ (X(v) \land \forall w (X(w) \to v \leq w)) \right]$$

**Fact (not proved here):** The set of sentences

$$\{ \varphi \} \cup \text{Th}(\mathcal{N}_1)$$

defines $\mathcal{N}_1$ (and every structure which is an expansion of $\mathcal{N}_1$) **up to isomorphism**, where $\mathcal{N}_1 \triangleq (\mathbb{N}, 0, S, <)$ in Handout 14.
A second-order sentence satisfied by a model $M$ iff the domain of $M$ is **infinite**:

$$\psi \triangleq \exists F \left[ \forall x \forall y \left( F(x) = F(y) \rightarrow x = y \right) \land \exists z \forall x \neg \left( F(x) = z \right) \right]$$

A second-order sentence satisfied by a model $M$ iff the domain of $M$ is **finite**:

$$\neg \psi$$
Compactness Theorem for First-Order (Handout 12). Let $\Gamma$ be a set of first-order sentences.

1. If every finite subset of $\Gamma$ is **satisfiable**, then so is $\Gamma$.
2. If every finite subset of $\Gamma$ is **consistent**, then so is $\Gamma$.

Counter-Example for Second-Order Compactness
For every $n \geq 1$, define the first-order sentence $\theta_n$ by:

$$\theta_n \triangleq \text{“there are at least } n \text{ distinct elements”}$$

Consider the set of sentences:

$$\Delta = \{\neg \psi\} \cup \{\theta_1, \theta_2, \theta_3, \ldots\}$$

Every finite subset of $\Delta$ is **satisfiable**, while $\Delta$ is **unsatisfiable**.