Model Checking: Translating (Propositional) LTL into First-Order Logic

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consider FOL models \( \mathcal{M} \) over \( \mathbb{N} = \{0, 1, 2, \ldots\} \) with \( \mathcal{F} = \emptyset \) and \( \mathcal{R} = \{ \langle \rangle \} \cup \{ \text{propositional variables used as unary predicates} \} \)

(Sometimes this is called FOMLO = “First-Order Monadic Logic of Linear Order”)

translation function \([ \_ ](\_): \{\text{LTL formulas}\} \times \mathbb{N} \rightarrow \{\text{FOL formulas}\}\)

\[
\begin{align*}
[p](t) &= p(t) \\
[\varphi \land \psi](t) &= [\varphi](t) \land [\psi](t) \\
[\neg \varphi](t) &= \neg [\varphi](t) \\
[X \varphi](t) &= [\varphi](t + 1) \\
[F \varphi](t) &= \exists t' \ [ t' \geq t \land [\varphi](t') ] \\
[G \varphi](t) &= \forall t' \ [ t' \geq t \rightarrow [\varphi](t') ] \\
[\varphi U \psi](t) &= \exists t' \ [ t' \geq t \land [\psi](t') \land \forall t'' \ [ t \leq t'' < t' \rightarrow [\varphi](t'') ] ] \\
\end{align*}
\]
Theorem. Let $\mathcal{M}$ be a transition system, i.e., $\mathcal{M}$ is a LTL model, and $\pi$ a path in $\mathcal{M}$. The following are equivalent assertions, for every WFF $\varphi$ of LTL and every $i \geq 0$ (not $i \geq 1$ as in the book):

1. $\pi^i \models_{\text{LTL}} \varphi$
2. there is a FOL model $\mathcal{N}$ such that $\mathcal{N} \models_{\text{FOL}} [\varphi](i)$

where $\mathcal{N}$ is over the vocabulary $\mathcal{F} = \emptyset$ and
\[ R = \{<\} \cup \{\text{propositional variables used as unary predicates}\} \]

Corollary. The following are equivalent, for every WFF $\varphi$ of LTL:

1. $\models_{\text{LTL}} \varphi$, \quad \text{i.e., $\varphi$ is semantically valid in LTL.}
2. $\models_{\text{FOL}} \forall t \ (\llbracket \varphi \rrbracket(t))$, \quad i.e., $\forall t \ (\llbracket \varphi \rrbracket(t))$ is semantically valid in FOL.
Question:
Is there a translation in the opposite direction, from FOL to LTL?

More precisely, is it the case that for every WFF $\varphi$ of “first-order monadic logic of linear order” we can define a WFF $\psi$ of LTL such that:

$\varphi$ is semantically valid iff $\psi$ is semantically valid?

Answer:
YES, by Kamp’s Theorem.