the CTL model-checking algorithm

Reading: [LCS, 3.6.1]

- the basic algorithm for CTL model-checking processes the given CTL WFF $\varphi$ (one of the two inputs) “from the inside out”, i.e., it starts from the smallest sub-WWF’s and works outwards towards $\varphi$.
- pseudo-code for the basic CTL model-checking algorithm is shown in [LCS, page 227].
- the state-explosion problem, [LCS, page 229]: e.g., adding a new propositional atom $p$ doubles the complexity of verifying a property involving $p$. 
an example using the CTL model-checking algorithm

INPUT:

1. \( \varphi \triangleq \text{AX EF } \psi \)
   where
   \( \psi \triangleq (\neg p \land q \land \neg r) \lor (p \land \neg q \land r) \)

2. transition system \( \mathcal{M} \) (on the right, start states not specified)
an example using the CTL model-checking algorithm

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   where
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2. transition system $\mathcal{M}$ (on the right, start states not specified)

   blue states are where sub-WFF: $(\neg p \land q \land \neg r)$ is satisfied
an example using the CTL model-checking algorithm

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   \((p \land \neg q \land r)\) is satisfied
an example using the CTL model-checking algorithm

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an example using the CTL model-checking algorithm

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red states are where sub-WFF: (EF \( \psi \)) is satisfied
an example using the CTL model-checking algorithm

INPUT:
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**double-red states** are where sub-WFF: \( \text{AX (EF } \psi) \) is satisfied

➤ **Conclusion:** \( \mathcal{M} \models \varphi \) iff \( s_6 \) is one of the start states of \( \mathcal{M} \)
more on the CTL model-checking algorithm

1. **Exercise:** Determine the states of $M$ (as defined in the preceding pages) satisfying the following CTL WFF’s:

   $\varphi_1 \triangleq \text{EG } q$
   $\varphi_2 \triangleq \text{A}(p \text{ U } \varphi_1) = \text{A}(p \text{ U } (\text{EG } q))$
   $\varphi_3 \triangleq \text{EX } \varphi_2 = \text{EX}(\text{A}(p \text{ U } (\text{EG } q)))$

2. **Problem** – illustrating abstraction (next slide) to alleviate state-explosion:
   For an arbitrary CTL WFF $\varphi$ and an arbitrary transition system $M$, let $M[\varphi]$ be the transition system obtained as follows:
   - For every state $s$, if $M, s \not\models \varphi$, then delete state $s$ and all transitions (i.e., edges) to $s$ and all transitions from $s$.

Prove the following statement:

- For a state $s$ in transition system $M$ and CTL WFF $\varphi$, we have that $M, s \models \text{EG } \varphi$ iff two conditions:
  
  (i) $M, s \models \varphi$ and
  (ii) there is a strongly-connected component of $M[\varphi]$ with at least one transition (i.e., edge) which is reachable from $s$. 
dealing with the state-explosion problem

**Reading:** [LCS, pp 229-230]. Different approaches:

- **Efficient data structures**: ordered binary decision diagrams (OBDD’s), which represent sets of states instead of individual states, studied in [LCS, Chapter 6].

- **Abstraction**: More “abstract” models, i.e., building transition systems with fewer details or no details affecting satisfaction of the WFF to be checked.

- **Partial order reduction**: For asynchronous systems, several interleaving of component “traces” (see lecture notes of March 24, and again in this handout from slide 26 and on) may be equivalent for the satisfaction of the WFF to be checked.

- **Induction**: Model-checking systems with “large” numbers of identical, or similar, components can often be implemented by some sort of induction on this number.

- **Composition**: Break the verification problem down into several simpler verification problems... (more in lecture).
dealing with the state-explosion problem

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- **Composition**: Break the verification problem down into several simpler verification problems . . . (more in lecture).
the LTL model-checking algorithm

Reading: [LCS, 3.6.3, pp 232-238] – highly condensed, more difficult to read.
minimal (and different) presentation of LTL

\[ \phi, \psi ::= \top | p | \neg \phi | \phi \land \psi | X \phi | \phi U \psi \]
minimal (and different) presentation of LTL

- syntax of LTL (compare with Handout 18, page 3):

$$\varphi, \psi ::= \top | p | \neg \varphi | \varphi \land \psi | X \varphi | \varphi U \psi$$
minimal (and different) presentation of LTL

- let \( p \) range over a set \( \text{AP} \) of atomic propositions \( \text{AP} = \{p_1, \ldots, p_n\} \)
minimal (and different) presentation of LTL

- let $p$ range over a set $AP$ of atomic propositions $AP = \{p_1, \ldots, p_n\}$

- let $2^{AP}$ be the set of all truth-value assignments to $AP$, i.e.,

$$2^{AP} = \left\{ (A(p_1), \ldots, A(p_n)) \mid A : AP \rightarrow \{\text{true}, \text{false}\} \right\}$$
minimal (and different) presentation of LTL

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  **Warning:** There are uncountably many $\omega$-words over $2^{AP}$ (why?), whereas there are only countably many non-deterministic Büchi automata (NBA) (why?). So, there are necessarily $\omega$-words over $2^{AP}$ that are not defined by NBA.
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- **Notational convention:** If $\sigma = A_0 A_1 A_2 \cdots$, then for all $0 \leq i \leq j$,

\[
\sigma[i] = A_i \quad \sigma[i..] = A_i A_{i+1} A_{i+2} \cdots \quad \sigma[i..j] = A_i A_{i+1} \cdots A_j
\]
minimal (and different) presentation of LTL

- semantics of LTL (compare with Handout 18, page 4 and on) – NO mention of transition systems so far, the interpretation is with respect to an $\omega$-word $\sigma \in (2^{AP})^\omega$:
minimal (and different) presentation of LTL

- semantics of LTL (compare with Handout 18, page 4 and on) – **NO** mention of transition systems so far, the interpretation is with respect to an $\omega$-word $\sigma \in \left(2^{\text{AP}}\right)^\omega$:

1. $\sigma \models T$

2. $\sigma \models p$ iff $p \in \sigma[0]$, i.e., $\sigma[0] \models p$

3. $\sigma \models \neg \varphi$ iff $\sigma \not\models \varphi$

4. $\sigma \models \varphi \land \psi$ iff $\sigma \models \varphi$ and $\sigma \models \psi$

5. $\sigma \models X\varphi$ iff $\sigma[1\ldots] \models \varphi$

6. $\sigma \models \varphi \mathbf{U} \psi$ iff there is $j \geq 0$ such $\sigma[j\ldots] \models \psi$ and $\sigma[i\ldots] \not\models \varphi$ for every $0 \leq i < j$
minimal (and different) presentation of LTL

> interpretation of a WFF $\varphi$ of LTL as a subset of $(2^{AP})^\omega$:

$$\omega\text{-words}(\varphi) \triangleq \left\{ \sigma \in (2^{AP})^\omega \mid \sigma \models \varphi \right\}$$
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- we mention **transition systems** for the first time now . . .

  let $\mathcal{M} = (S, \rightarrow, L)$ be a transition system over AP
  
  let $\pi$ be an infinite execution path in $\mathcal{M}$
  
  let $\text{trace}(\pi)$ be the $\omega$-word in $(2^{\text{AP}})^\omega$ induced by $\pi$
minimal (and different) presentation of LTL

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  - $\pi \models \varphi$ iff $\text{trace}(\pi) \models \varphi$

  **Remark:** You can take this equivalence as a definition, or you can take it as relating the previous definition (Handout 18) to the current one, i.e.:

  $\pi \models_{\text{old}} \varphi$ iff $\text{trace}(\pi) \models_{\text{new}} \varphi$. 

Assaf Kfoury, CS 512, Spring 2016, Handout 23
minimal (and different) presentation of LTL

- interpretation of a WFF $\varphi$ of LTL as a subset of $(2^{AP})^\omega$:
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  \omega\text{-words}(\varphi) \triangleq \left\{ \sigma \in (2^{AP})^\omega \mid \sigma \models \varphi \right\}
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- $s \models \varphi$ iff $\pi \models \varphi$ for every execution path $\pi$ that starts at $s$
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- $M \models \varphi \iff \text{Traces}(M) \subseteq \omega\text{-words}(\varphi)$
model-checking algorithm for LTL

- basic idea is based on the following observation:

- for an arbitrary transition system $\mathcal{M}$ and an arbitrary WFF $\varphi$ of LTL

$$\mathcal{M} \models \varphi \quad \text{iff} \quad \text{Traces}(\mathcal{M}) \subseteq \omega\text{-words}(\varphi)$$

$$\text{iff} \quad \text{Traces}(\mathcal{M}) \cap \left( (2^{\text{AP}})^\omega - \omega\text{-words}(\varphi) \right) = \emptyset$$

$$\text{iff} \quad \text{Traces}(\mathcal{M}) \cap \omega\text{-words}(\neg \varphi) = \emptyset$$
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$$\iff \text{Traces}(M) \cap \omega\text{-words}(\neg \varphi) = \emptyset$$

- hence, if we can construct non-deterministic Buchi automaton $A$ s.t.

$$L_\omega(A) = \omega\text{-words}(\neg \varphi)$$

then $M \models \varphi \iff \text{Traces}(M) \cap L_\omega(A) = \emptyset$
model-checking algorithm for LTL

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- hence, if we can construct non-deterministic Buchi automaton $A$ s.t.

\[
\mathcal{L}_\omega(A) = \omega\text{-words}(\neg \varphi)
\]

then $\mathcal{M} \models \varphi$ iff $\text{Traces}(\mathcal{M}) \cap \mathcal{L}_\omega(A) = \emptyset$

**Remark:** Strictly speaking, the equality $\mathcal{L}_\omega(A) = \omega\text{-words}(\neg \varphi)$ cannot hold (why?). For it to hold, we need to restrict $\omega\text{-words}(\neg \varphi)$ to a $\omega$-regular subset . . . .
implementation of model-checking algorithm for LTL

two major steps:
implementation of model-checking algorithm for LTL

two major steps:

▶ for an arbitrary LTL WFF $\varphi$, construct an NBA $A$ representing $\neg \varphi$, i.e., construct $A$ over the alphabet $2^{AP}$ such that the $\omega$-regular language accepted by $A$ is precisely the set $\omega$-words($\neg \varphi$):

$$\mathcal{L}_\omega(A) = \omega\text{-words}(\neg \varphi)$$

http://www.lsv.ens-cachan.fr/~gastin/ltl2ba/index.php
implementation of model-checking algorithm for LTL

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\[
\mathcal{L}_\omega(\mathcal{A}) = \omega\text{-words}(\neg \varphi)
\]

- construct the “product” NBA \( \mathcal{B} \) and determine whether \( \mathcal{L}_\omega(\mathcal{B}) = \emptyset \):

\[
\mathcal{B} \triangleq \mathcal{M} \otimes \mathcal{A}
\]

http://www.lsv.ens-cachan.fr/~gastin/ltl2ba/index.php
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$$L_\omega(A) = \omega\text{-words}(\neg \varphi)$$

▶ construct the “product” NBA $B$ and determine whether $L_\omega(B) = \emptyset$:

$$B \triangleq M \otimes A$$

▶ **IF** there is an $\omega$-word accepted by $B$,

**THEN** $\text{Traces}(M) \cap L_\omega(A) \neq \emptyset$ and $M \not\models \varphi$,

**ELSE** $\text{Traces}(M) \cap L_\omega(A) = \emptyset$ and $M \models \varphi$