Predicate logic: Soundness and Completeness, Definability and Expressiveness

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Finishing Up Modeling

○ Modeling the winning strategy for Tic-tac-toe:
  \[ \exists M_0 \forall M_1 \exists M_2 \forall M_3 \ldots \forall M_{K^2} \text{ } X \text{ wins} \]
  where the dimensions of the board are \( K \times K \) (i.e. there are a max of \( K^2 \) moves) and \( M_i \) is the \( i^{th} \) move.

First Order Logic: So Far

○ So far, we have looked at formal syntax, formal proof rules, and formal semantics

○ Prenex form, which is useful for QBFs

○ Reminder: Propositional logic is a subset of both QBF and first-order logic, but QBF is not a subset of first-order logic (contains other WFFs)

First Order Logic: Soundness and Completeness

Consistency

○ Definition: Let \( \Gamma \) be a set of WFFs. We say \( \Gamma \) is consistent if \( \Gamma \nvdash \bot \)

○ Definition: A sentence is a closed WFF

○ The following are equivalent:
  1. \( \Gamma \) is consistent
  2. There is no WFF \( \varphi \) such that \( \Gamma \vdash \varphi \) and \( \Gamma \vdash \lnot \varphi \)
  3. There exists a WFF \( \varphi \) such that \( \Gamma \nvdash \varphi \)

○ Similarly, the following are also equivalent:
  1. \( \Gamma \) is inconsistent
  2. There is a WFF \( \varphi \) such that \( \Gamma \vdash \varphi \) and \( \Gamma \vdash \lnot \varphi \)
  3. For all WFF \( \varphi \), \( \Gamma \vdash \varphi \)

○ Theorem: Let \( \Gamma \) be a set and \( \varphi \) be a WFF. Then \( \Gamma \cup \{ \varphi \} \) is consistent iff \( \Gamma \nvdash \lnot \varphi \). Similarly, \( \Gamma \cup \{ \varphi \} \) is inconsistent iff \( \Gamma \vdash \lnot \varphi \)

Soundness
Soundness means $\Gamma \vdash \varphi \rightarrow \Gamma \models \varphi$ ("If you can prove it, then it’s true."). The proof for this is in the book section 2.3.

Corollary: If $\Gamma$ is satisfiable (i.e. there exists an assignment of variables such that every WFF in $\Gamma$ evaluates to true), then $\Gamma$ is consistent.

- Note that while the soundness theorem goes from syntax to semantics, the corollary goes from semantics to syntax
- Proof of Corollary (by contrapositive):
  Suppose $\Gamma$ is inconsistent. Then there exists a WFF $\varphi$ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$. Then, by soundness, we have $\Gamma \models \varphi$ and $\Gamma \models \neg \varphi$, so $\Gamma \models \varphi \land \neg \varphi = \bot$. Then $\Gamma \models \bot$, so $\Gamma$ is not satisfiable.

Completeness

- **Theorem:** Let $\Gamma$ be a set of sentences. If $\Gamma$ is consistent, then $\Gamma$ is satisfiable.
  - The proof for this uses the Model-existence lemma.

- **Corollary:** If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.
  - **Proof Sketch:**
    Every interpretation of $\Gamma$ is a model of $\Gamma$, so you can use $\Gamma$ as premises to deduce $\varphi$.

Undecidability of Soundness and Completeness

- Recall: a WFF $\varphi$ is *valid* if it is true for all possible valuations, i.e. a tautology.
- **Theorem:** It is undecidable whether a first-order WFF is semantically valid (or, by soundness and completeness, formally provable). That is, there is no decision procedure $A$ such that, given an arbitrary first-order WFF $\varphi$:
  1. $A$ will always terminate
  2. $A$ will always return yes if $\varphi$ is valid, no if $\varphi$ is not valid
  - This is one of the canonical problems of P-space complexity
  - The proof for this relies on reducing the halting problem

Compactness

- **Theorem:** Let $\Gamma$ be a set of first-order WFFs. If every finite subset of $\Gamma$ is satisfiable, then $\Gamma$ is satisfiable. Similarly, if every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.
- **Theorem (Koenig’s Lemma):** Let $T$ be a (possibly infinite) binary tree. If, for every $n \geq 1$ there exists a path of length $n$ in $T$, then $T$ has infinite depth, i.e. there exists a path of infinite length.
  - This holds for every $T$ of bounded node-degree
  - Suppose $T$ has unbounded node-degree?
  * Consider the tree $T = \langle T_0, < T_1^1, < T_2^1, T_2^2 >, < T_3^1, T_3^2, T_3^3 > ... > \rangle$, i.e. a tree where the first child of the root is a single vertex, the second child is a straight path of two vertices, the third child is a straight path of three vertices, etc. Then for every $n \geq 1$ there exists a path of length $n$ in $T$, but $T$ does not have infinite depth.
Reachability is not first-order definable: There does not exist a first order WFF $\psi(x,y)$ with two free variables $x, y$ over the signature $(R,=)$ where $R$ is a binary predicate such that for every graph model $M = (M, R^M, =)$ and every $a, b \in M$, $M \models \psi[a, b]$ iff there is a path from $a$ to $b$.

**Notation**

Let $M$ be a model, $\ell$ be a look-up table, $\varphi$ be a first-order WFF such that $M, \ell \models \varphi$

- If $\varphi$ is closed, we can simply write $M \models \varphi$, that is, for all $\ell$, $M, \ell \models \varphi$
- If $\varphi$ has free variables $x_1, x_2, x_3, \ldots$ and $\ell(x_i) = a_i$, the following are equivalent notations
  1. $M, a_1, a_2, a_3, \ldots \models \varphi(x_1, x_2, x_3, \ldots)$
  2. $M \models \varphi[a_1, a_2, a_3, \ldots]$
  3. $(M, a_1, a_2, a_3, \ldots) \models \varphi[a_1/x_1][a_2/x_2][a_3/x_3]...$