CS 512, Spring 2016, Handout 02

Natural Deduction, and Examples of Natural Deduction, in Propositional Logic

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January 19, 2017

► IF the train arrives late AND there are NO taxis

THEN John is late for the meeting

- ► IF the train arrives late AND there are NO taxis

 THEN John is late for the meeting
- John is NOT late for the meeting

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 THEN John is late for the meeting
- John is NOT late for the meeting
- the train did arrive late

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again symbolically:

▶ IF P AND (NOT Q) THEN R

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- John is NOT late for the meeting
- the train did arrive late
- ► THEREFORE there were taxis

$$\qquad \qquad (\quad P \quad \land \quad \neg Q \quad) \rightarrow \quad R$$

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- $ightharpoonup (P \land \neg Q) \rightarrow R$
- $ightharpoonup \neg R$

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- $ightharpoonup \neg R$
- \triangleright P
- $\vdash Q$

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again symbolically:

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- \triangleright P
- \vdash Q

more succintly:

$$P \wedge \neg Q \rightarrow R$$
, $\neg R$, $P \vdash Q$



a sequent (also called a judgment) is an expression of the form:

$$\varphi_1,\ldots,\varphi_n \vdash \psi$$

where:

- 1. $\varphi_1, \ldots, \varphi_n, \psi$ are well-formed formulas (also called wff's)
- 2. the symbol "\—" is pronounced turnstile
- 3. the wff's $\varphi_1, \dots, \varphi_n$ to the left of " \vdash " are called the **premises** (also called **antecedents** or **hypotheses**)
- 4. the wff ψ to the right of " \vdash " is called the **conclusion** (also called **succedent**)

- a sequent is said to be valid (also deducible or derivable) if there is a formal proof for it
- a formal proof (also called deduction or derivation) is a sequence of wff's which starts with the premises of the sequent and finishes with the conclusion of the sequent:

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arphi_1 premise arphi_2 premise arphi_n premise arphi_n premise arphi conclusion
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where every wff in the deduction is obtained from the wff's preceding it using a proof rule

$$\qquad \qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \wedge i$$

$$\qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \wedge \mathbf{i}$$

$$-\frac{\varphi \wedge \psi}{\varphi} \wedge e_1$$

$$\qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \wedge \mathbf{i}$$

$$ightharpoonup \left(\frac{\varphi \wedge \psi}{\varphi} \right) \wedge \mathsf{e}_1$$

$$\frac{\varphi \wedge \psi}{\psi} \wedge e_2$$

$$\qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \quad \wedge \mathbf{i}$$

$$\qquad \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \wedge e_1$$

$$-\frac{\varphi \wedge \psi}{\psi}$$
 $\wedge e_2$

$$\rightarrow \frac{\varphi}{\neg \neg \varphi}$$

$$\qquad \qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \wedge i$$

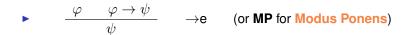
$$\qquad \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \wedge \mathsf{e}_1$$

$$-\frac{\varphi \wedge \psi}{\psi}$$
 $\wedge e_2$

$$\frac{\varphi}{\neg \neg \varphi}$$
 $\neg \neg |$

$$-\frac{\neg\neg\varphi}{\varphi}$$
 $\neg\neg\epsilon$

(cannot be used in intuitionistic logic)



$$\qquad \qquad \frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \qquad \rightarrow \text{e} \qquad \text{(or MP for Modus Ponens)}$$

$$\qquad \qquad \frac{\varphi \to \psi \qquad \neg \psi}{\neg \varphi} \qquad \text{MT} \qquad \text{(for Modus Tollens)}$$

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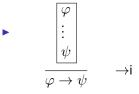
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$$\begin{array}{ccc} & & & & & \\ & \varphi & & & \\ \vdots & & & \\ \psi & & & \\ \hline \varphi \to \psi & & \to \mathbf{i} \end{array}$$

open a box when you *introduce* an **assumption** (wff φ in rule \rightarrow i)

$$\qquad \qquad \frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \qquad \rightarrow \text{e} \qquad \text{(or MP for Modus Ponens)}$$

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open a box when you *introduce* an assumption (wff φ in rule \rightarrow i) close the box when you *discharge* the assumption

open a box when you *introduce* an assumption (wff φ in rule \to i) close the box when you *discharge* the assumption you must close every box and discharge every assumption in order to complete a formal proof

Proof Rules Associated with Only One "¬" and with "⊥"

So far, we have an **elimination** rule and an **introduction** rule for double negation " $\neg\neg$ ", namely $\neg\neg$ e and $\neg\neg$ i, but not for single negation " \neg ". We now compensate for this lack:

$$ho$$
 $\frac{\varphi}{\varphi}$ $\neg \varphi$ $\neg \varphi$ (or **LNC** for **Law of Non-Contradiction**)

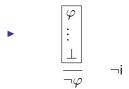
where "\perp " (a single symbol) stands for "contradiction"

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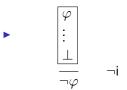
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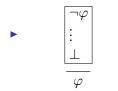


$$\frac{\perp}{\varphi}$$
 \perp e ("if you can prove \perp , you can prove every WFF")

Two Derived Proof Rules

The two following rules are derived rules -

the first from rules \rightarrow i, \neg i, \rightarrow e, and $\neg\neg$ e (see [LCS, pp 24-25]); the second from rules \vee i, \neg i, \neg e, and $\neg\neg$ e (see [LCS, pp 25-26]):



PBC (for Proof by Contradiction)

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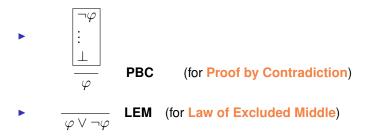


Because ¬¬e is rejected in intuitionistic logic, so are PBC and LEM

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(a summary of all proof rules and some derived rules in [LCS, p. 27])

formal proof of the sequent $P \vdash Q \rightarrow (P \land Q)$

formal proof of the sequent $P \vdash Q \rightarrow (P \land Q)$

$$_3$$
 $P \wedge Q$

$$\wedge$$
i 1, 2

$$_4$$
 $Q \rightarrow (P \land Q)$

$$\rightarrow$$
i

formal proof of the sequent $P \to (Q \to R) \vdash P \land Q \to R$

formal proof of the sequent $P o (Q o R) dash P \wedge Q o R$

$$_1$$
 $P o (Q o R)$

$_{2}$ $P \wedge Q$	
3 P	$\wedge e_1$ 2
$_4$ $Q o R$	ightarrowe 1,3
5 Q	$\wedge e_2$ 2
6 R	ightarrowe 4,5
$_{7}$ $P \wedge Q \rightarrow R$	ightarrowi

formal proof of the sequent $P \land Q \rightarrow R \vdash P \rightarrow (Q \rightarrow R)$

formal proof of the sequent $P \land Q \rightarrow R \vdash P \rightarrow (Q \rightarrow R)$

$$_1$$
 $P \wedge Q \rightarrow R$

2 P	
3 Q	
$_4$ $P \wedge Q$	∧i 2, 3
5 R	ightarrowe 1,4
6 $Q o R$	ightarrowi

$$_{7}$$
 $P \rightarrow (Q \rightarrow R)$ $\rightarrow i$

formal proof of the sequent $P o (Q o R) \vdash (P o Q) o (P o R)$

formal proof of the sequent $P \to (Q \to R) \vdash (P \to Q) \to (P \to R)$

$$_1$$
 $P \rightarrow (Q \rightarrow R)$

$_{2}$ $P \rightarrow Q$	
3 P	
4 <i>Q</i>	\rightarrow e 2,3
$_{5}$ $Q \rightarrow R$	→e 1,3
6 R	ightarrowe $5,4$
$_7$ $P o R$	ightarrowi

 $8 (P \rightarrow Q) \rightarrow (P \rightarrow R)$

Formal Proof of the Initial Sequent:

► Initial Sequent

premise

- $_{1}$ $P \wedge \neg Q \rightarrow R$
- $_{2}$ $\neg R$ premise
- ₃ P premise
- $_{4}$ $\neg Q$ assume
- $_{5}$ $P \wedge \neg Q$ \wedge i 3,4
- $_{6}$ R \rightarrow e 1,5
 - ¬e 6,2
 - $\neg Q$
- $_{9}$ $_{Q}$