CS 512, Spring 2016, Handout 02

Natural Deduction, and Examples of Natural Deduction, in Propositional Logic

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from **informal/common** reasoning to **formal** reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  THEN John is late for the meeting

from **informal/common** reasoning to **formal** reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  THEN John is late for the meeting
- John is **NOT** late for the meeting

from **informal/common** reasoning to **formal** reasoning:

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- the train did arrive late

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- **THEREFORE** there were taxis

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again symbolically:

- **IF** P **AND** (**NOT** Q) **THEN** R

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again symbolically:

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- $\neg R$
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more succintly:

$$P \wedge \neg Q \rightarrow R,\ \ \neg R,\ \ P\ \vdash\ Q$$

Formal Proof of the Sequent $* * *$

- a **sequent** (also called a **judgment**) is an expression of the form:

$$\varphi_1, \ldots, \varphi_n \;\vdash\; \psi$$

where:

1. $\varphi_1, \ldots, \varphi_n, \psi$ are **well-formed formulas** (also called **wff's**)
2. the symbol "$\vdash$" is pronounced **turnstile**
3. the wff's $\varphi_1, \ldots, \varphi_n$ to the left of "$\vdash$" are called the **premises** (also called **antecedents** or **hypotheses**)
4. the wff ψ to the right of "$\vdash$" is called the **conclusion** (also called **succedent**)

- a sequent is said to be **valid** (also **deducible** or **derivable**) if there is a **formal proof** for it

- a **formal proof** (also called **deduction** or **derivation**) is a sequence of wff's which starts with the **premises** of the sequent and finishes with the **conclusion** of the sequent:

φ_1	premise
φ_2	premise
$\vdots$	
φ_n	premise
$\vdots$	
ψ	conclusion

  where every wff in the deduction is obtained from the wff's preceding it using a **proof rule**

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- $$\frac{\varphi \wedge \psi}{\psi} \quad \wedge\mathrm{e}_2$$
- $$\frac{\varphi}{\neg\neg\varphi} \quad \neg\neg\mathrm{i}$$
- $$\frac{\neg\neg\varphi}{\varphi} \quad \neg\neg\mathrm{e}$$ (cannot be used in **intuitionistic logic**)

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open a box when you *introduce* an **assumption** (wff φ in rule $\to$i)

close the box when you *discharge* the **assumption**

you must close every **box** and discharge every **assumption**
in order to complete a formal proof

Proof Rules Associated with Only One “¬” and with “⊥”

So far, we have an **elimination** rule and an **introduction** rule for double negation “¬¬”, namely ¬¬e and ¬¬i, but not for single negation “¬”. We now compensate for this lack:

- $$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg\text{e}$$ (or **LNC** for **Law of Non-Contradiction**)

  where “⊥” (a single symbol) stands for “**contradiction**”

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- $$\frac{\boxed{\begin{array}{c}\varphi \\ \vdots \\ \bot\end{array}}}{\neg\varphi} \quad \neg\text{i}$$

- $$\frac{\bot}{\varphi} \quad \bot\text{e}$$ (“if you can prove ⊥, you can prove every WFF”)

Two Derived Proof Rules

The two following rules are derived rules –
the first from rules $\rightarrow$i, $\neg$i, $\rightarrow$e, and $\neg\neg$e (see [LCS, pp 24-25]);
the second from rules $\vee$i, $\neg$i, $\neg$e, and $\neg\neg$e (see [LCS, pp 25-26]):

- $$\dfrac{\boxed{\begin{array}{l}\neg\varphi\\ \vdots\\ \bot\end{array}}}{\varphi}$$ **PBC** (for **Proof by Contradiction**)

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Because $\neg\neg$e is rejected in **intuitionistic logic**, so are **PBC** and **LEM**

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(a summary of all **proof rules** and some **derived rules** in [LCS, p. 27])

Examples of Natural Deductions

formal proof of the sequent $P \vdash Q \rightarrow (P \wedge Q)$

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1	P	
2	Q	
3	$P \wedge Q$	$\wedge$i $1, 2$
4	$Q \rightarrow (P \wedge Q)$	$\rightarrow$i

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1	$P \to (Q \to R)$	
2	$P \land Q$	
3	P	$\land e_1$ 2
4	$Q \to R$	$\to e$ 1, 3
5	Q	$\land e_2$ 2
6	R	$\to e$ 4, 5
7	$P \land Q \to R$	$\to i$

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2	P	
3	Q	
4	$P \wedge Q$	$\wedge$i 2, 3
5	R	$\to$e 1, 4
6	$Q \to R$	$\to$i
7	$P \to (Q \to R)$	$\to$i

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2	$P \to Q$	
3	P	
4	Q	$\to$e $2, 3$
5	$Q \to R$	$\to$e $1, 3$
6	R	$\to$e $5, 4$
7	$P \to R$	$\to$i
8	$(P \to Q) \to (P \to R)$	$\to$i

Formal Proof of the Initial Sequent:

▸ Initial Sequent

1	$P \wedge \neg Q \rightarrow R$	premise
2	$\neg R$	premise
3	P	premise
4	$\neg Q$	assume
5	$P \wedge \neg Q$	$\wedge$i $3, 4$
6	R	$\rightarrow$e $1, 5$
7	$\bot$	$\neg$e $6, 2$
8	$\neg\neg Q$	$\neg$i
9	Q	$\neg\neg$e 8