CS 512, Spring 2017, Handout 03 Semantics of *Classical* Propositional Logic

(as opposed to *Intuitionistic* Propositional Logic)

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some terminology

- the semantics (or formal semantics) of a formal logic L is sometimes called the model theory of L.
- ► the model theory of classical propositional logic is defined in terms of Boolean algebras: a model (or interpretation) for the logic is a two-element Boolean algebra, *i.e.*, an assignment of truth-values to the propositional atoms with the standard boolean operations on them (∧, ∨, and ¬).
- the standard boolean operations can be defined using truth tables.
- the model theory of intuitionistic propositional logic can be defined in terms of Heyting algebras (also called pseudo-Boolean algebras): a model (or interpretation) is a Heyting algebra.
- ► every Heyting algebra satisfying the law of excluded middle a ∨ ¬a = ⊤ or, equivalently, the double negation law ¬¬a = a is a Boolean algebra.

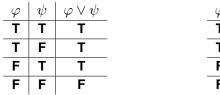
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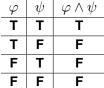
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introductory remarks

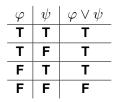
- for the semantics of <u>classical</u> propositional logic, it suffices to consider the familiar **two-element Boolean algebra**.
- the two-element Boolean algebra is only one member of the infinite family of Boolean algebras (for more on this topic, click here).
- the two-element Boolean algebra is not the only way of defining the semantics of propositional logic, *e.g.*, we can use what are called three-valued Kleene algebras to define the semantics of propositional logic (click here).
- Heyting algebras is not the only way of defining the semantics of intuitionistic propositional logic, *e.g.*, we can use what are called Kripke structures instead (click here and here).

logical "or" (\lor) and logical "and" (\land)



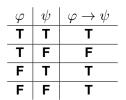


logical "or" (\lor) and logical "and" (\land)

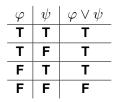


φ	ψ	$\varphi \wedge \psi$
Т	Т	Т
Т	F	F
F	Т	F

logical "implication" (\rightarrow)

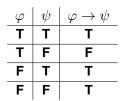


logical "or" (\lor) and logical "and" (\land)



φ	ψ	$\varphi \wedge \psi$
Т	Т	Т
Т	F	F
F	Т	F
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logical "implication" (\rightarrow)



and similarly for "negation" (\neg) and many other logical connectives

- start with all the propositional atoms in the WFF
- incrementally, consider each sub-WFF, from innermost to outermost



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Р	Q	$\neg P$
Т	Т	F
Т	F	F
F	Т	Т
F	F	Т

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Р	Q	$\neg P$	$\neg Q$
Т	Т	F	F
Т	F	F	Т
F	Т	Т	F
F	F	Т	Т

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Ρ	Q T	$\neg P$ F	$\neg Q$ F	$P ightarrow \neg Q$
		F	F	F
Т	F	F	T	Т
F	Т	T	F	Т
F	F	Т	Т	Т

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P	Q	$\neg P$	$\neg Q$	$P ightarrow \neg Q$	$Q \lor \neg P$	$(P \to \neg Q) \to (Q \lor \neg P)$ \mathbf{T}
Т	Т	F	F	F	Т	т
Т	F	F	Т	Т	F	F
F	Т	Т	F	T T	Т	т
F	F	Т	Т	Т	Т	Т

- propositional WFF φ is satisfiable if there is an assignment of truth-values to the propositional atoms which makes φ true.
- propositional WFF φ is a tautology if every assignment of truth-values to the propositional atoms makes φ true.

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Т	F	F	Т	Т	F	F
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- propositional WFF φ is satisfiable if there is an assignment of truth-values to the propositional atoms which makes φ true.
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- ▶ $(P \rightarrow \neg Q) \rightarrow (Q \lor \neg P)$ is satisfiable, but is not a tautology.

related to the sequent $(P \land \neg Q) \rightarrow R, \ \neg R, \ P \vdash Q$

shown valid, *i.e.*, formally derivable by the proof rules at the end of Handout 02.

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P	Q	R	$\neg Q$	$\neg R$	$P \wedge \neg Q$	$(P \land \neg Q) \to R$
Т	Т	Τ	F	F	F	Т
Т	Т	F	F	Т	F	т
Т	F	Т	Т	F	Т	Т
Т	F	F	Т	Т	Т	F
F	Т	Τ	F	F	F	Т
F	Т	F	F	Т	F	Т
F	F	Т	Т	F	F	Т
F	F	F	Т	Т	F	Т

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shown valid, *i.e.*, formally derivable by the proof rules at the end of Handout 02.

P	Q	R	$ \neg Q$	$\neg R$	$P \wedge \neg Q$	$(P \land \neg Q) \to R$
Т	Т	Т	F	F	F	Т
Т	Т	F	F	Т	F	Т
Т	F	Т	Т	F	Т	Т
Т	F	F	Т	Т	Т	F
F	Т	T	F	F	F	Т
F	Т	F	F	Т	F	Т
F	F	Τ	Т	F	F	Т
F	F	F	Т	Т	F	Т

when all the premises (shaded in gray) evaluate to T, so does the conclusion (shaded in green)

related to the sequent $(P \land \neg Q) \rightarrow R, \ \neg R, \ P \vdash Q$

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P	Q	R	$\neg Q$	$\neg R$	$P \wedge \neg Q$	$(P \land \neg Q) \to R$
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Т	Т	F	F	Т	F	т
Т	F	Т	Т	F	Т	Т
Т	F	F	Т	Т	Т	F
F	Т	Τ	F	F	F	Т
F	Т	F	F	Т	F	Т
F	F	Τ	Т	F	F	Т
F	F	F	Т	Т	F	Т

- when all the premises (shaded in gray) evaluate to T, so does the conclusion (shaded in green)
- ▶ in such a case we write $(P \land \neg Q) \rightarrow R, \neg R, P \models Q$

If, for every interpretation/model/valuation

 (*i.e.*, assignment of truth values to the propositional atoms)
 for which all of the WFF's φ₁, φ₂, ..., φ_n evaluate to T,
 it is also the case that ψ evaluates to T, then we write:

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$$

and say that " $\varphi_1, \varphi_2, \ldots, \varphi_n$ semantically entails ψ " or also "every model of $\varphi_1, \varphi_2, \ldots, \varphi_n$ is a model of ψ ".

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Theorem (Soundness):

If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.

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Theorem (Soundness):

If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.

Theorem (Completeness):

If $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$.

▶ simple version of **soundness**: if $\vdash \psi$ then $\models \psi$

Informally, "if you can prove it, then it is true".

▶ simple version of **completeness**: if $\models \psi$ then $\vdash \psi$

Informally, "if it is true, then you can prove it".

- if $\models \psi$, then we say ψ is a **tautology**.
- if $\vdash \varphi$, then we say φ is **valid** or a (formal) theorem.