CS 512, Spring 2017, Handout 03

Semantics of *Classical* Propositional Logic

(as opposed to *Intuitionistic* Propositional Logic)

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some terminology

- the **semantics** (or **formal semantics**) of a formal logic $\mathcal{L}$ is sometimes called the **model theory** of $\mathcal{L}$.
- the **model theory** of **classical propositional logic** is defined in terms of **Boolean algebras**: a **model** (or **interpretation**) for the logic is a two-element Boolean algebra, *i.e.*, an assignment of truth-values to the propositional atoms with the standard boolean operations on them ($\wedge$, $\vee$, and $\neg$).
- the standard boolean operations can be defined using **truth tables**.
- the **model theory** of **intuitionistic propositional logic** can be defined in terms of **Heyting algebras** (also called **pseudo-Boolean algebras**): a **model** (or **interpretation**) is a Heyting algebra.
- every **Heyting algebra** satisfying the **law of excluded middle** $a \vee \neg a = \top$ or, equivalently, the **double negation law** $\neg\neg a = a$ is a **Boolean algebra**.

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- the **model theory** of **intuitionistic propositional logic** can be defined in terms of **Heyting algebras** (also called **pseudo-Boolean algebras**): a **model** (or **interpretation**) is a Heyting algebra.
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introductory remarks

- for the semantics of classical propositional logic, it suffices to consider the familiar **two-element Boolean algebra**.
- the two-element Boolean algebra is only one member of the infinite family of Boolean algebras (for more on this topic, click here).
- the two-element Boolean algebra is not the only way of defining the semantics of propositional logic, *e.g.*, we can use what are called **three-valued Kleene algebras** to define the semantics of propositional logic (click here).
- Heyting algebras is not the only way of defining the semantics of intuitionistic propositional logic, *e.g.*, we can use what are called **Kripke structures** instead (click here and here).

some familiar truth-tables:

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logical “or” ($\vee$) and logical “and” ($\wedge$)

φ	ψ	$\varphi \vee \psi$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

φ	ψ	$\varphi \wedge \psi$
T	**T**	**T**
T	**F**	**F**
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logical "implication" ($\rightarrow$)

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T	**F**	**F**
F	**T**	**T**
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F	**T**	**F**
F	**F**	**F**

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φ	ψ	$\varphi \rightarrow \psi$
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T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

and similarly for "negation" ($\neg$) and many other logical connectives . . . .

a more complicated truth-table

for propositional WFF $(P \to \neg Q) \to (Q \vee \neg P)$:

- start with all the propositional atoms in the WFF
- incrementally, consider each sub-WFF, from innermost to outermost

P	Q
T	**T**
T	**F**
F	**T**
F	**F**

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T	**T**	**F**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

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T	**T**	**F**	**F**
T	**F**	**F**	**T**
F	**T**	**T**	**F**
F	**F**	**T**	**T**

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P	Q	$\neg P$	$\neg Q$	$P \to \neg Q$
T	**T**	**F**	**F**	**F**
T	**F**	**F**	**T**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

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T	**T**	**F**	**F**	**F**	**T**
T	**F**	**F**	**T**	**T**	**F**
F	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**

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P	Q	$\neg P$	$\neg Q$	$P \to \neg Q$	$Q \vee \neg P$	$(P \to \neg Q) \to (Q \vee \neg P)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

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P	Q	$\neg P$	$\neg Q$	$P \to \neg Q$	$Q \vee \neg P$	$(P \to \neg Q) \to (Q \vee \neg P)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

- propositional WFF φ is **satisfiable** if **there is** an assignment of truth-values to the propositional atoms which makes φ true.
- propositional WFF φ is a **tautology** if **every** assignment of truth-values to the propositional atoms makes φ true.

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for propositional WFF $(P \to \neg Q) \to (Q \vee \neg P)$:

- start with all the propositional atoms in the WFF
- incrementally, consider each sub-WFF, from innermost to outermost

P	Q	$\neg P$	$\neg Q$	$P \to \neg Q$	$Q \vee \neg P$	$(P \to \neg Q) \to (Q \vee \neg P)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

- propositional WFF φ is **satisfiable** if **there is** an assignment of truth-values to the propositional atoms which makes φ true.
- propositional WFF φ is a **tautology** if **every** assignment of truth-values to the propositional atoms makes φ true.
- $(P \to \neg Q) \to (Q \vee \neg P)$ is satisfiable, but is not a tautology.

another more complicated truth-table

related to the sequent $(P \wedge \neg Q) \rightarrow R,\ \neg R,\ P \vdash Q$

shown **valid**, *i.e.*, formally derivable by the proof rules at the end of **Handout 02**.

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P	Q	R	$\neg Q$	$\neg R$	$P \wedge \neg Q$	$(P \wedge \neg Q) \to R$
T	**T**	**T**	**F**	**F**	**F**	**T**
T	**T**	**F**	**F**	**T**	**F**	**T**
T	**F**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**T**	**F**
F	**T**	**T**	**F**	**F**	**F**	**T**
F	**T**	**F**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**F**	**F**	**T**
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related to the sequent $(P \wedge \neg Q) \to R,\ \neg R,\ P \vdash Q$
shown **valid**, *i.e.*, formally derivable by the proof rules at the end of **Handout 02**.

P	Q	R	$\neg Q$	$\neg R$	$P \wedge \neg Q$	$(P \wedge \neg Q) \to R$
T	**T**	**T**	**F**	**F**	**F**	**T**
T	**T**	**F**	**F**	**T**	**F**	**T**
T	**F**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**T**	**F**
F	**T**	**T**	**F**	**F**	**F**	**T**
F	**T**	**F**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**F**	**F**	**T**
F	**F**	**F**	**T**	**T**	**F**	**T**

- when all the premises (shaded in gray) evaluate to **T**, so does the conclusion (shaded in green)

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related to the sequent $(P \wedge \neg Q) \to R,\ \neg R,\ P \vdash Q$
shown **valid**, *i.e.*, formally derivable by the proof rules at the end of **Handout 02**.

P	Q	R	$\neg Q$	$\neg R$	$P \wedge \neg Q$	$(P \wedge \neg Q) \to R$
T	T	T	F	F	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	T	F	F	T	F	T
F	F	T	T	F	F	T
F	F	F	T	T	F	T

- when all the premises (shaded in gray) evaluate to **T**, so does the conclusion (shaded in green)
- in such a case we write $(P \wedge \neg Q) \to R,\ \neg R,\ P \models Q$

soundness and **completeness**

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- If, for every interpretation/model/valuation (*i.e.*, assignment of truth values to the propositional atoms) for which all of the WFF's $\varphi_1, \varphi_2, \ldots, \varphi_n$ evaluate to **T**, it is also the case that ψ evaluates to **T**, then we write:

  $$\varphi_1,\ \varphi_2,\ \ldots\ ,\ \varphi_n\ \models\ \psi$$

  and say that "$\varphi_1, \varphi_2, \ldots, \varphi_n$ semantically entails ψ"
  or also "every model of $\varphi_1, \varphi_2, \ldots, \varphi_n$ is a model of ψ".

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- **Theorem (Soundness):**
  If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.

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- **Theorem (Completeness):**
  If $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$.

soundness and **completeness**

- simple version of **soundness**: if $\vdash \psi$ then $\models \psi$

  Informally, "if you can prove it, then it is true".

- simple version of **completeness**: if $\models \psi$ then $\vdash \psi$

  Informally, "if it is true, then you can prove it".

- if $\models \psi$, then we say ψ is a **tautology**.

- if $\vdash \varphi$, then we say φ is **valid** or a **(formal) theorem**.