CS 512, Spring 2017, Handout 05

Semantics of *Classical* Propositional Logic

(Continued)

Soundness, Completeness, Compactness

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soundness

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- Let Γ a (possibly infinite) set of propositional WFF's.

  If, for every model/interpretation/valuation
  (*i.e.*, assignment of truth values to prop atoms), it holds that:

  - whenever all the WFF's in Γ evaluate to **T**,
  - it is also the case that ψ evaluates to **T**,

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- **Theorem (Soundness):** If $\Gamma \vdash \psi$ then $\Gamma \models \psi$.

  (Slightly stronger than the statement of Soundness in [LCS, Theorem 1.35, p 46]:
  If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.)

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- **Proof idea:** "Course-of-values" induction on $n \geqslant 1$
  (in a later **Handout 06**).

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- If Γ is infinite, we need another preliminary result: Compactness.

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- For proofs of Compactness above and its corollary, click here.