CS 512, Spring 2017, Handout 06

Classical Propositional Logic: Proof Sketch of its Soundness

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soundness once more

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  Induction hypothesis (IH): Soundness holds for every $k' < k$.

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$\vdots$	$\vdots$	
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- Use IH on $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi_{k_1}$ and $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi_{k_2}$.