CS 512, Spring 2017, Handout 09

Analytic Tableaux

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Origins and background

- **semantic tableaux** were invented by Evert Beth (1908-1964) in the mid-1950's, later adapted and simplified and called **analytic tableaux** by Raymond Smullyan (1919-2017) in the 1960's.
- henceforth, we use exclusively Smullyan's formulation (only a small part of it) and say **tableaux** (singular: *tableau*, plural: *tableaux*) instead of "analytic tableaux".
- if you are interested in making connections with formal proof systems, the tableaux method can be viewed as minor variation of the so-called **Gentzen's cut-free sequent calculus** (click here for more details).
- the tableaux method has proved to be easily adapted as a procedure to decide the **validity** and **satisfiability** of WFF's in several formal logics (*e.g.*, **modal logics** as developed by Jaakko Hintikka (1929-2015) and Saul Kripke (1940-)) beside propositional logic and first-order logic.
- this handout deals exclusively with the tableaux method as a decision procedure for **classical propositional logic**, though our presentation is easily extended to closely related **intuitionistic** propositional logic and first-order logic.
- we present a simple version of the method, perhaps the most basic, as there are several variations with more complicated notational conventions.

Informal comments and comparison with natural deduction

- a **tableau** is a finite tree growing downward (root node at the top), where every non-leaf node has one or two successor nodes.
- a **tableau** satisfies the subformula property, *i.e.*, every node in the tree is a subformula of the formula at the root.
- a **tableau** can be viewed as a formal deduction turned upside-down, *e.g.*, whereas in a **natural-deduction** proof we start from the premises at the top and conclude with the full WFF at the bottom, in a **tableau** we start from the full WFF at the top and decompose it into its subformulas as we proceed downward.
- a **tableau** requires *less* guess-work, *less* search for the next rule to apply, and *less* creativity than a **natural-deduction** proof (to prove the validity of the same WFF), *i.e.*, the former (a tableau) is more mechanical than the latter (a natural deduction) and more easily implemented as an algorithm.

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(the preceding comments in support of the **tableaux** method apply again when we compare it to other formal proof systems beside **natural deduction**, *e.g.*, any of the **Hilbert-style** systems.)

expansion rules of the tableaux method

- There are several versions of the **tableaux method** for classical propositional logic, and each is easily extended to classical first-order logic (in a later handout).
- Differences between these versions are mostly minor syntactic (which may nevertheless have an impact, good or bad, on their computer implementations).
- One important version omitted here (for lack of time) uses **signed expansion rules**, *i.e.*, rules where antecedents and conclusions are signed with **T** or **F**.

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- Here, we present what is arguably the simplest version of the **tableaux method** for classical PL, which involves seven **(unsigned) expansion rules**:

$$\frac{\varphi \wedge \psi}{\begin{array}{l}\varphi \\ \psi\end{array}} \qquad \frac{\neg(\varphi \wedge \psi)}{\neg\varphi \mid \neg\psi} \qquad \frac{\varphi \vee \psi}{\varphi \mid \psi} \qquad \frac{\neg(\varphi \vee \psi)}{\begin{array}{l}\neg\varphi \\ \neg\psi\end{array}}$$

$$\frac{\varphi \to \psi}{\neg\varphi \mid \psi} \qquad \frac{\neg(\varphi \to \psi)}{\begin{array}{l}\varphi \\ \neg\psi\end{array}} \qquad \frac{\neg\neg\varphi}{\varphi}$$

- We illustrate the application of these rules on a few examples before we define **tableaux method** in full generality.

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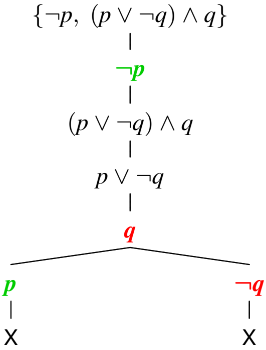
- We illustrate the application of these rules on a few examples before we define **tableaux method** in full generality.
- (If you want to see some of the other versions of the **tableaux method**, click here and consult some of the references mentioned therein.)

examples of the tableaux method

We can apply the tableaux method to a single WFF or to a finite set of WFF's. Here, we apply it to the set $\Gamma = \{\neg p,\ (p \vee \neg q) \wedge q\}$:



EXPANSION RULES

$$\frac{\varphi \wedge \psi}{\begin{array}{l}\varphi \\ \psi\end{array}}$$

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$$\{\neg p,\ (p \vee \neg q) \wedge q\}$$

$$\neg p$$

$$(p \vee \neg q) \wedge q$$

$$p \vee \neg q$$

$$q$$

$$p \qquad\qquad\qquad \neg q$$

$$\text{X} \qquad\qquad\qquad \text{X}$$

There are two paths from the root node (at the top) to the leaf nodes (at the bottom). Both paths are **closed** (marked by X), because each includes a propositional variable and its negation (p along the left path, q along the right path). We conclude that Γ is **unsatisfiable**.

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examples of the tableaux method

We show de Morgan's law $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$ is valid (a tautology) by showing its negation is a contradiction (unsatisfiable) :

$$\neg(\neg(p \wedge q) \rightarrow (\neg p \vee \neg q))$$

$$\neg(p \wedge q)$$

$$\neg(\neg p \vee \neg q)$$

$$\neg\neg p$$

$$\neg\neg q$$

$$p$$

$$q$$

$$\begin{array}{cc} \neg p & \neg q \\ \mathrm{X} & \mathrm{X} \end{array}$$

Compare the tableau above with the **natural-deduction proof** (page 7 of Handout 8) and the **truth table** (page 12 of Handout 8).

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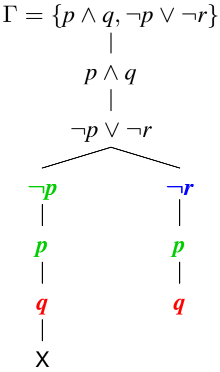
examples of the tableaux method

We apply the tableaux method to the set $\Gamma = \{p \wedge q, \neg p \vee \neg r\}$:



Only the left path is closed, the right path is open. All the rules have been applied to all the WFF's along the same path. Hence, the tableau cannot be closed, which implies the initial set Γ is satisfiable. From the right path we obtain a Boolean valuation σ satisfying Γ: $\sigma(p) = \mathbf{T}$, $\sigma(q) = \mathbf{T}$, $\sigma(r) = \mathbf{F}$.

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examples of the tableaux method

- For more examples of how to apply the tableaux method to decide *validity* or *satisfiability* of propositional WFF's, search the Web.
- Try, for example, the following website: *Propositional tableaux*, click here . Consider, in particular, the heuristics that are proposed in that website to improve performance.

definitions of analytic tableaux

- A **tableau** is a finite unordered tree, with *root node* at the top and *leaf nodes* at the bottom, where every node is labelled with a WFF.
- Let $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ be a finite set of WFF's. The tree with a single path:

  $$\begin{array}{c} \varphi_1 \\ | \\ \vdots \\ | \\ \varphi_n \end{array}$$

  is a tableau for Γ.
- If T is a tableau for a set Γ of WFF's, and T' is obtained from T by applying an expansion rule, then T' is a tableau for Γ.

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- If T is a tableau for a set Γ of WFF's, and T' is obtained from T by applying an expansion rule, then T' is a tableau for Γ.

- A path from the root to a leaf in tableau T is a **closed path** if it includes both a WFF ψ and its negation $\neg\psi$. Otherwise the path is said to be an **open path**.
- A tableau T is a **closed tableau** if all its paths are closed.
- A **tableau proof** for a single WFF φ is a closed tableau for the singleton $\{\neg\varphi\}$.

more definitions of analytic tableaux

- A path π in a tableau is a **maximal path** if, for every non-atomic WFF φ occurring in π, there is a node of π below φ where the expansion rule for φ is applied.
- A tableau T is a **maximal tableau** if every path in T is closed or maximal.
- A tableau T is a **strict tableau** if, for every WFF φ, the expansion rule for φ is applied at most once on every path containing φ. But note that the same WFF φ in two distinct paths π_1 and π_2 can be used independently, *i.e.*, the expansion rule for φ can be used at most once for π_1 and at most once for π_2.
- In some accounts of the tableaux method elsewhere, you will find the following definition: A tableau T is a **regular tableau** if, on no path of T, a WFF appears more than once. However, for our choice of expansion rules in this handout, this definition of "regular tableau" is too restrictive.

  Exercise: Show that if a tableau T is regular, then T is strict.

  Exercise: Define a WFF φ for which no closed regular tableau exists, though there is a closed strict tableau for φ. *Hint*: Consider $\varphi = (p \wedge q) \wedge (p \wedge \neg q)$.

basic theorems about analytic tableaux

Let Γ be a finite set of propositional WFF's and $\varphi = \bigwedge \Gamma$. Recall:

- Γ is unsatisfiable $\Leftrightarrow$ φ is a contradiction,
- φ is a contradiction $\Leftrightarrow$ the last column in the truth-table of φ is all **F**'s,
- in such a case, we write $\Gamma \models \perp$ or, equivalently, $\varphi \models \perp$.

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- Γ is unsatisfiable $\Leftrightarrow$ φ is a contradiction,
- φ is a contradiction $\Leftrightarrow$ the last column in the truth-table of φ is all **F**'s,
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Theorem (Refutation Completeness of Tableaux)

For a finite set Γ of propositional WFF's, if $\Gamma \models \bot$, then there is a closed tableau for Γ.

Theorem (Soundness of Tableaux)

For a finite set Γ of propositional WFF's, if there is a closed tableau for Γ, then $\Gamma \models \bot$.

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- We do **not** say **completeness** of tableaux, but **refutation completeness** of tableaux, because the tableaux method does not provide a set of *rewrite/expansion rules* to confirm a *semantic entailment* of the form $\Gamma \models \psi$ unless $\psi = \bot$.
- However, **refutation completeness** is **not** a fundamental limitation, because:

  $\Gamma \models \psi \quad \Leftrightarrow \quad \Gamma \cup \{\neg\psi\}$ is unsatisfiable $\quad \Leftrightarrow \quad (\bigwedge \Gamma) \wedge \neg\psi$ is a contradiction.

  Hence, tableaux can be used to establish *semantic entailment* $\Gamma \models \psi$ in general.

  (For a proof of these equivalences, see Lemma 6 in *Compactness of Propositional and First-Order Logic* – click here .)

basic theorems about analytic tableaux

Theorem

For a finite set Γ of propositional WFF's:

- Γ *is satisfiable iff there is a maximal tableau for* Γ *with an open path.*
- Γ *is satisfiable iff there exists no closed strict tableau for* Γ*.*

Corollary

To decide whether a WFF φ is valid/a tautology, it suffices to construct a strict maximal closed tableau for $\{\neg\varphi\}$.

Exercise As a formal proof system for classical propositional logic, prove the following:

1. refutation-completeness of the tableaux method,
2. soundness of the tableaux method.

Hint: These are easy!

Exercise Let T be a maximal or closed tableau for a finite set Γ of WFF's. Show that if T is strict, then T is necessarily finite.

Hint: If T is a finitely branching tree and every path in T is finite, then T is finite.

tableaux vs. *natural-deduction proofs* vs. *truth-tables*

(Continuation of the discussion in the second half of Handout 08.)

Exercise

1. Use the **tableaux method** to show the validity of the following more general version of de Morgan's law (1):

$$\varphi_1 \;\triangleq\; \neg(p \wedge q \wedge r) \;\rightarrow\; (\neg p \vee \neg q \vee \neg r)$$

   Hint: Use the tableau on slide 10 as a guide.

2. Use the **tableaux method** to show the validity of de Morgan's law (1) in general:

$$\varphi_2 \;\triangleq\; \neg(p_1 \wedge \cdots \wedge p_n) \;\rightarrow\; (\neg p_1 \vee \cdots \vee \neg p_n)$$

   where $n \geqslant 2$.

3. Compute the precise size of the tableau, in Part 2 above, as a function of n. Add up the total number of symbols appearing in the tableau (without including all matching parentheses).

4. Compare the complexity of the **tableau proof** for φ_2 in Part 2 above with the complexity of the **natural-deduction proof** of φ_2 and that of the **truth-table** verification of φ_2. For the latter two procedures, consult Handout 08.

tableaux vs. *natural-deduction proofs* vs. *truth-tables*

The preceding exercise shows that to prove the validity of some propositional WFF's, such as the general de Morgan's law (1), the **tableau method** is a clear winner. However, it **cannot** be the winner to prove all valid propositional WFF's (**Why?**).

tableaux vs. *natural-deduction proofs* vs. *truth-tables*

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Exercise

1. We compare **tableaux** and **truth-tables** relative to the same initial WFF φ. The complexity of one primarily depends on (a) the number of *occurrences of logical connectives* in φ, whereas the complexity of the other primarily depends on (b) the number of *distinct propositional variables* in φ. Which method depends on (a) and which on (b)? Explain carefully.

2. Let $X = \{x_1, \ldots, x_k\}$ be a set of $k \geqslant 1$ variables. We define 2^k distinct WFF's $\varphi_1, \ldots, \varphi_{2^k}$ where each φ_j is a disjunction containing x_i or $\neg x_i$ for every $1 \leqslant i \leqslant k$. The conjunction Ψ of these WFF's, *i.e.*, $\Psi = \bigwedge\{\varphi_1, \ldots, \varphi_{2^k}\}$, is not satisfiable. Give a precise argument for each of the following:

   2.1 A closed tableau for Ψ contains at least $k!$ distinct paths.

   2.2 A truth-table analysis of Ψ can be carried out in $\mathcal{O}(k^2 \cdot 2^{2k})$ steps. *Hint*: Ψ contains $\mathcal{O}(k \cdot 2^k)$ occurrences of variables and operators.

   2.3 Truth-tables are more efficient than tableaux on WFF's such as Ψ (an instance of a *conjunctive normal form*). *Hint*: $k! >>> k^2 \cdot 2^{2k}$.