Propositional Logic:
Conjunctive Normal Forms,
Disjunctive Normal Forms,
Horn Formulas,
and other special forms

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conjunctive normal form & disjunctive normal form
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CNF

\[ L ::= p \mid \neg p \quad \text{literal} \]
\[ D ::= L \mid L \lor D \quad \text{disjunction of literals} \]
\[ C ::= D \mid D \land C \quad \text{conjunction of disjunctions} \]
**conjunctive normal form & disjunctive normal form**

**CNF**

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\begin{align*}
L & ::= p \mid \neg p & \text{literal} \\
D & ::= L \mid L \lor D & \text{disjunction of literals} \\
C & ::= D \mid D \land C & \text{conjunction of disjunctions}
\end{align*}
\]

**DNF**

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\begin{align*}
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C & ::= L \mid L \land C & \text{conjunction of literals} \\
D & ::= C \mid C \lor D & \text{disjunction of conjunctions}
\end{align*}
\]
Why CNF?

A disjunction of literals $L_1 \lor \cdots \lor L_m$ is valid (equivalently, is a tautology) iff there are $1 \leq i, j \leq m$ with $i \neq j$ such that $L_i$ is $\neg L_j$.

A conjunction of disjunctions $D_1 \land \cdots \land D_n$ is valid (equivalently, is a tautology) iff for every $1 \leq i \leq n$ it is the case that $D_i$ is valid.

CNF allows for a fast and easy syntactic test of validity.

Unfortunately, conversion into CNF may lead to exponential blow-up:

$$(x_1 \land y_1) \lor (x_2 \land y_2) \lor \cdots \lor (x_n \land y_n)$$

becomes

$$(x_1 \lor \cdots \lor x_{n-1} \lor x_n) \land (x_1 \lor \cdots \lor x_{n-1} \lor y_n) \land \cdots \land (y_1 \lor \cdots \lor y_{n-1} \lor y_n)$$

i.e., the initial WFF of size $O(n)$ becomes an equivalent WFF of size $O(2^n)$, because each clause in the latter contains either $x_i$ or $y_i$ for every $i$.

Converting a WFF into an equivalent WFF in CNF, preserving validity, is NP-hard!

(However, converting a WFF into another WFF, not necessarily equivalent, preserving satisfiability can be carried out in linear time – more in a later handout.)
Why CNF?

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- A conjunction of disjunctions \( D_1 \land \cdots \land D_n \) is valid (equivalently, is a tautology) iff for every \( 1 \leq i \leq n \) it is the case that \( D_i \) is valid.

\[ (x_1 \land x_2 \land \cdots \land x_n) \lor (y_1 \land y_2 \land \cdots \land y_n) \]

which becomes

\[ (x_1 \lor \cdots \lor x_{n-1} \lor x_n) \land (x_1 \lor \cdots \lor x_{n-1} \lor y_n) \land \cdots \land (y_1 \lor \cdots \lor y_{n-1} \lor y_n) \]

\[ \text{i.e., the initial WFF of size } O(n) \text{ becomes an equivalent WFF of size } O(2^n), \text{ because each clause in the latter contains either } x_i \text{ or } y_i \text{ for every } i. \]

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becomes

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i.e., the initial WFF of size $O(n)$ becomes an equivalent WFF of size $O(2^n)$, because each clause in the latter contains either $x_i$ or $y_i$ for every $i$.

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Why CNF?

- A disjunction of literals \( L_1 \lor \cdots \lor L_m \) is **valid** (equivalently, is a **tautology**) iff there are \( 1 \leq i, j \leq m \) with \( i \neq j \) such that \( L_i \) is \( \neg L_j \).

- A conjunction of disjunctions \( D_1 \land \cdots \land D_n \) is **valid** (equivalently, is a **tautology**) iff for every \( 1 \leq i \leq n \) it is the case that \( D_i \) is valid.

- **CNF** allows for a fast and easy syntactic test of **validity**.

- Unfortunately, conversion into **CNF** may lead to exponential blow-up:

\[
(x_1 \land y_1) \lor (x_2 \land y_2) \lor \cdots \lor (x_n \land y_n) \ \text{becomes} \\
(x_1 \lor \cdots \lor x_{n-1} \lor x_n) \land (x_1 \lor \cdots \lor x_{n-1} \lor y_n) \land \cdots \land (y_1 \lor \cdots \lor y_{n-1} \lor y_n)
\]

i.e., the initial WFF of size \( \mathcal{O}(n) \) becomes an equivalent WFF of size \( \mathcal{O}(2^n) \), because each clause in the latter contains either \( x_i \) or \( y_i \) for every \( i \).

- Converting a WFF into an equivalent WFF in **CNF**, preserving **validity**, is NP-hard!

(However, converting a WFF into another WFF, not necessarily equivalent, preserving **satisfiability** can be carried out in linear time – more in a later handout.)
Why DNF?

A conjunction of literals $L_1 \land \cdots \land L_m$ is satisfiable iff \{ $L_1, \ldots, L_m$ \} does not include both a propositional atom $P$ and its negation $\neg P$.

A disjunction of conjunctions $C_1 \lor \cdots \lor C_n$ is satisfiable iff there is some $1 \leq i \leq n$ such that $C_i$ is satisfiable.

DNF allows for a fast and easy syntactic test of satisfiability.

Unfortunately, conversion into DNF may lead to exponential blow-up: $(x_1 \lor y_1) \land (x_2 \lor y_2) \land \cdots \land (x_n \lor y_n)$ becomes $(x_1 \land \cdots \land x_{n-1} \land x_n) \lor (x_1 \land \cdots \land x_{n-1} \land y_n) \lor \cdots \lor (y_1 \land \cdots \land y_{n-1} \land y_n)$, i.e., the initial WFF of size $O(n)$ becomes an equivalent WFF of size $O(2^n)$, because each clause in the latter contains either $x_i$ or $y_i$ for every $i$.

Converting a WFF into an equivalent WFF in DNF, preserving satisfiability, is NP-hard!
Why DNF?

- A conjunction of literals $L_1 \land \cdots \land L_m$ is \textbf{satisfiable} iff \{$L_1, \ldots, L_m$\} does not include both a propositional atom $P$ and its negation $\neg P$.

- \textbf{DNF} allows for a fast and easy syntactic test of satisfiability.

- Unfortunately, conversion into \textbf{DNF} may lead to exponential blow-up: $(x_1 \lor y_1) \land (x_2 \lor y_2) \land \cdots \land (x_n \lor y_n)$ becomes $(x_1 \land \cdots \land x_{n-1} \land x_n) \lor (x_1 \land \cdots \land x_{n-1} \land y_n) \lor \cdots \lor (y_1 \land \cdots \land y_{n-1} \land y_n)$, i.e., the initial WFF of size $O(n)$ becomes an equivalent WFF of size $O(2^n)$, because each clause in the latter contains either $x_i$ or $y_i$ for every $i$.

- Converting a WFF into an equivalent WFF in \textbf{DNF}, preserving satisfiability, is NP-hard!
Why DNF?

- A conjunction of literals $L_1 \land \cdots \land L_m$ is **satisfiable** iff $\{L_1, \ldots, L_m\}$ does not include both a propositional atom $P$ and its negation $\neg P$.

- A disjunction of conjunctions $C_1 \lor \cdots \lor C_n$ is **satisfiable** iff there is some $1 \leq i \leq n$ such that $C_i$ is satisfiable.
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▶ A disjunction of conjunctions $C_1 \lor \cdots \lor C_n$ is **satisfiable** iff there is some $1 \leq i \leq n$ such that $C_i$ is satisfiable.

▶ **DNF** allows for a fast and easy syntactic test of **satisfiability**.
Why DNF?

- A conjunction of literals \( L_1 \land \cdots \land L_m \) is **satisfiable** iff \( \{ L_1, \ldots, L_m \} \) does not include both a propositional atom \( P \) and its negation \( \neg P \).

- A disjunction of conjunctions \( C_1 \lor \cdots \lor C_n \) is **satisfiable** iff there is some \( 1 \leq i \leq n \) such that \( C_i \) is satisfiable.

- **DNF** allows for a fast and easy syntactic test of **satisfiability**.

- Unfortunately, conversion into **DNF** may lead to exponential blow-up:

\[
(x_1 \lor y_1) \land (x_2 \lor y_2) \land \cdots \land (x_n \lor y_n) \text{ becomes } \\
(x_1 \land \cdots \land x_{n-1} \land x_n) \lor (x_1 \land \cdots \land x_{n-1} \land y_n) \lor \cdots \lor (y_1 \land \cdots \land y_{n-1} \land y_n)
\]

i.e., the initial WFF of size \( O(n) \) becomes an equivalent WFF of size \( O(2^n) \), because each clause in the latter contains either \( x_i \) or \( y_i \) for every \( i \).

- Converting a WFF into an equivalent WFF in **DNF**, preserving **satisfiability**, is NP-hard!
further comments on CNF and DNF, summing up:

- propositional WFF’s can be partitioned into three disjoint subsets:
  1. tautologies, or **unfalsifiable** WFF’s
  2. contradictions, or **unsatisfiable** WFF’s
  3. WFF’s that are both **satisfiable** and **falsifiable**

- satisfiability of:
  - **CNF** is in NP
  - **DNF** is in P

- tautology of:
  - **CNF** is in P
  - **DNF** is in co-NP

- falsifiability of:
  - **CNF** is in P
  - **DNF** is in NP
other special forms of propositional WFF’s:

- One such form is that of the WFF’s in **negation normal form (NNF)**: the negation operator ($\neg$) is only applied to variables, and the only logical operators are conjunction ($\wedge$) and disjunction ($\vee$).
other special forms of propositional WFF’s:

▶ One such form is that of the WFF’s in **negation normal form (NNF)**: the negation operator (¬) is only applied to variables, and the only logical operators are conjunction (∧) and disjunction (∨).

▶ More formally:

\[
L ::= \ p \ | \ \neg p \\
\varphi ::= \ L \ | \ \varphi \land \psi \ | \ \varphi \lor \psi
\]

▶ Fact: Every WFF in CNF or in DNF is also in NNF, but the converse is not true in general. See next slide for an example.

▶ Fact: NNF is not a canonical form, in contrast to CNF and DNF.

Example: 

\[\begin{align*}
&x \land (y \lor \neg z) \\
&\quad \text{and} \\
&\quad (x \land y) \lor (x \land \neg z)
\end{align*}\]

are equivalent and both in NNF.

▶ Fact: Every propositional WFF \( \varphi \) can be translated in linear time into an equivalent propositional WFF \( \psi \) in NNF such that 

\[|\psi| < \left( \frac{3}{2} \right) \cdot |\varphi|\]

Proof. Left to you.
other special forms of propositional WFF’s:

▶ One such form is that of the WFF’s in **negation normal form (NNF)**: the negation operator ($\neg$) is only applied to variables, and the only logical operators are conjunction ($\land$) and disjunction ($\lor$).

▶ More formally:

\[
L ::= p \mid \neg p \\
\varphi ::= L \mid \varphi \land \psi \mid \varphi \lor \psi
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▶ **Fact**: Every WFF in **CNF** or in **DNF** is also in **NNF**, but the converse is not true in general. See next slide for an example.

▶ **Fact**: **NNF** is not a canonical form, in contrast to **CNF** and **DNF**.

**Example**: $x \land (y \lor \neg z)$ and $(x \land y) \lor (x \land \neg z)$ are equivalent and both in **NNF**.

▶ **Fact**: Every propositional WFF $\varphi$ can be translated in linear time into an equivalent propositional WFF $\psi$ in **NNF** such that $|\psi| < (3/2) \cdot |\varphi|$.

**Proof**: Left to you.
example of a WFF in **NNF**, which is neither in **CNF** nor in **DNF**

\[
\left( (\neg p \land q) \lor (\neg q \land p) \right) \land \left( (r \land s) \lor (\neg s \land \neg r) \right)
\]

\[
\lor \left( (\neg p \land \neg q) \lor (q \land p) \right) \land \left( (r \land \neg s) \lor (s \land \neg r) \right)
\]
example of a WFF in **NNF**, which is neither in **CNF** nor in **DNF**

\[
\left( \left( \neg p \land q \right) \lor \left( \neg q \land p \right) \right) \land \left( \left( r \land s \right) \lor \left( \neg s \land \neg r \right) \right) \\
\lor \left( \left( \neg p \land \neg q \right) \lor \left( q \land p \right) \right) \land \left( \left( r \land \neg s \right) \lor \left( s \land \neg r \right) \right)
\]

and its **parse tree** after merging identical leaf nodes, turning it into a more compact **dag**:
another special form of propositional WFFs:

*Decomposable Negation Normal Form (DNNF)*
another special form of propositional WFFs: 

**Decomposable Negation Normal Form (DNNF)**

A propositional WFF $\varphi$ is a **decomposable negation normal form (DNNF)** if it is a NNF satisfying the **decomposability property**:

for every conjunction $\psi = \psi_1 \land \cdots \land \psi_n$ which is a sub-WFF of $\varphi$, no propositional variable/atom is shared by any two distinct conjuncts of $\psi$:

$$\text{Var}(\psi_i) \cap \text{Var}(\psi_j) = \emptyset \quad \text{for every} \quad i \neq j$$

Example: The NNF shown on page 19 is in fact a DNNF.

Fact: Satisfiability of WFF in DNNF is decidable in linear time.
another special form of propositional WFFs: 
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another special form of propositional WFFs:

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**Example**: The NNF shown on page 19 is in fact a DNNF.

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an important restricted class: *Horn formulas*

\[
P ::= \bot | \top | p
\]

\[
A ::= P \mid P \land A
\]

\[
C ::= A \rightarrow P
\]

Horn clause

\[
H ::= C \mid C \land H
\]

Horn formula

**Fact**: Satisfiability of Horn clauses is decidable in linear time.

**Proof**: To see this, rewrite a Horn clause into an equivalent disjunction of literals:

\[
L_1 \land \cdots \land L_n \rightarrow L \equiv \neg L_1 \lor \cdots \lor \neg L_n \lor L.
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**Fact**: Satisfiability of Horn formulas is decidable in linear time.
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C ::= A \rightarrow P \quad \text{Horn clause}
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H ::= C \mid C \land H \quad \text{Horn formula}
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\[ P ::= \bot \mid \top \mid p \]
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\[ C ::= A \rightarrow P \quad \text{Horn clause} \]
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**Fact:** Satisfiability of Horn formulas is decidable in linear time.
**Exercise** Search the Web to identify one or two applications, or areas of computer science, where each of the following forms are encountered:

1. Propositional WFF’s in **NNF**.
2. Propositional WFF’s in **DNNF**.
3. Propositional WFF’s that are **Horn** formulas.