CS 512, Spring 2017, Handout 15

Syntax of Predicate Logic (aka First-Order Logic)

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March 14, 2017

from English reasoning to formal reasoning:

for all x, **if** x is a bird **then** x has wings

for all x, **if** x has wings **then** x can fly

Coco is a bird

Coco has wings

Coco's *mother* can fly

from English reasoning to formal reasoning:

for all x, **if** x is a bird **then** x has wings	$\forall x\ (B(x)\ \rightarrow\ W(x))$
for all x, **if** x has wings **then** x can fly	$\forall x\ (W(x)\ \rightarrow\ F(x))$
Coco is a bird	$B(\mathbf{C})$
Coco has wings	$W(\mathbf{C})$
Coco's *mother* can fly	$F(m(\mathbf{C}))$

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it is **not** the case that **for all** x ...	$\neg(\forall x\ (B(x) \rightarrow W(x)))$
there exists an x such that ...	$\exists x\ (B(x) \wedge \neg W(x))$

WFF's of predicate logic

- *vocabulary* (a.k.a. *similarity type*, a.k.a. *signature*):

- *terms*:

- *well-formed formulas*:

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    $$t \ ::= \ x \mid c \mid f(t, \ldots, t)$$

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- are all WFF's of propositional logic WFF's of predicate logic ???

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but cannot **overlap**

free and **bound** variables (continued)

- the set of **free variables** in terms t and WFF's φ:

$$\mathsf{FV}(t) = \begin{cases} \varnothing & \text{if } t = c \\ \{x\} & \text{if } t = x \\ \mathsf{FV}(t_1) \cup \cdots \cup \mathsf{FV}(t_n) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

$$\mathsf{FV}(\varphi) = \begin{cases} \mathsf{FV}(t_1) \cup \cdots \cup \mathsf{FV}(t_n) & \text{if } \varphi = P(t_1, \ldots, t_n) \\ \mathsf{FV}(t_1) \cup \mathsf{FV}(t_2) & \text{if } \varphi = (t_1 \doteq t_2) \\ \mathsf{FV}(\varphi') & \text{if } \varphi = \neg\varphi' \\ \mathsf{FV}(\varphi_1) \cup \mathsf{FV}(\varphi_2) & \text{if } \varphi = (\varphi_1 \star \varphi_2),\ \star \in \{\wedge, \vee, \to\} \\ \mathsf{FV}(\varphi') - \{x\} & \text{if } \varphi = (\mathbf{Q}x\ \varphi') \text{ and } \mathbf{Q} \in \{\forall, \exists\} \end{cases}$$

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- **assumption**: every variable x has $\leqslant 1$ binding occurrence in any WFF (is this realistic?)

  this assumption is not essential, but without it, a variable may occur both free and bound in the same WFF.

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  every variable x has $\leqslant 1$ binding occurrence in any WFF,
- consider a WFF φ (**not** satisfying the **assumption**), say:

$$\varphi = \quad \cdots \Big(\mathbf{Q}_1\, x\, (\cdots\ x\ \cdots) \Big) \cdots \Big(\mathbf{Q}_2\, x\, (\cdots\ x\ \cdots) \Big) \cdots$$

  where $\mathbf{Q}_1, \mathbf{Q}_2 \in \{\forall, \exists\}$,
- is φ equivalent to:

$$\varphi' = \quad \cdots \Big(\mathbf{Q}_1\, x\, (\cdots\ x\ \cdots) \Big) \cdots \Big(\mathbf{Q}_2\ x'\ (\cdots\ x'\ \cdots) \cdots \Big)\ ??$$

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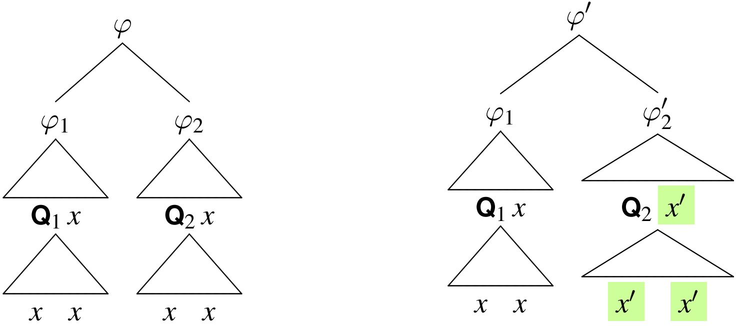
- yes, φ and φ' are equivalent

  Exercise: define the algorithm to transform φ into φ'

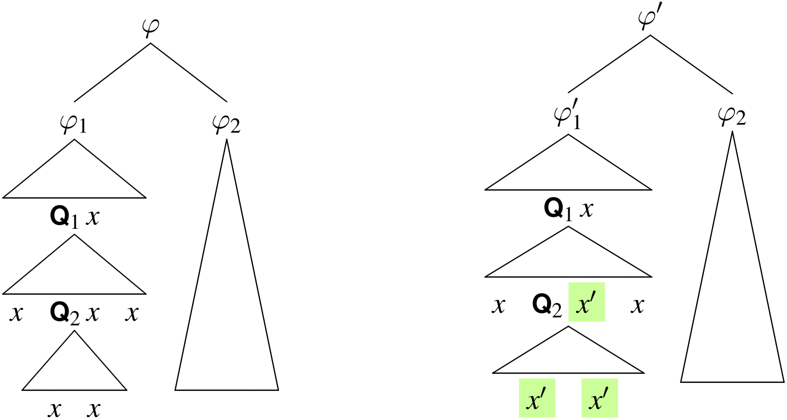
free and **bound** variables (continued)

renaming binding occurrences "$\mathbf{Q}_1\, x$" and "$\mathbf{Q}_2\, x$" in **disjoint** scopes



free and **bound** variables (continued)

renaming binding occurrences “$\mathbf{Q}_1\, x$” and “$\mathbf{Q}_2\, x$” in **nested** scopes



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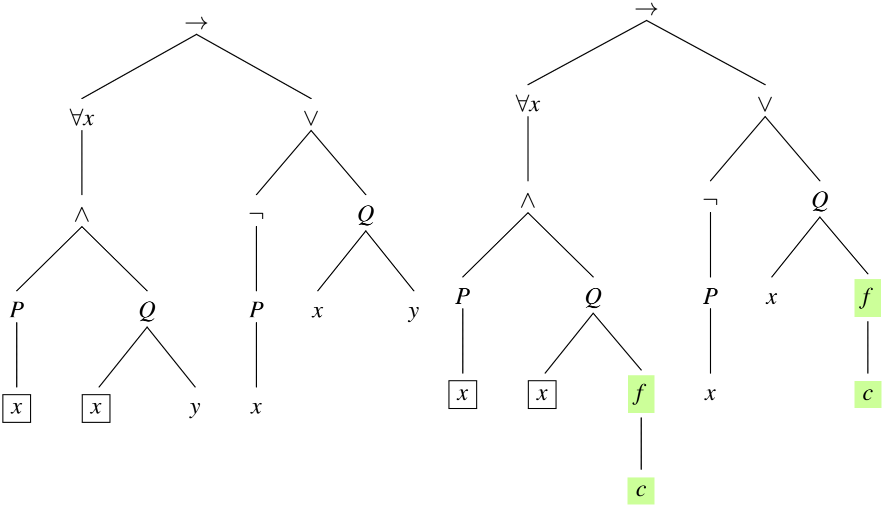

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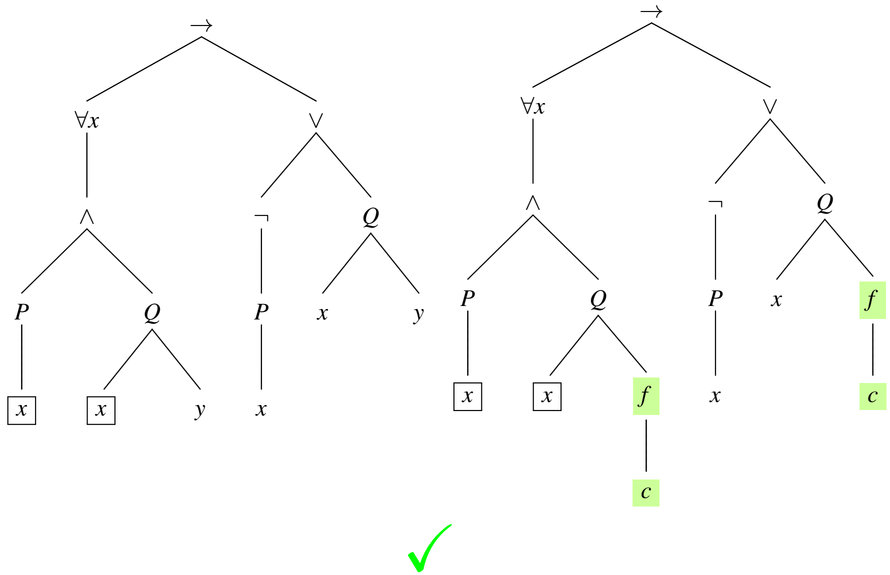

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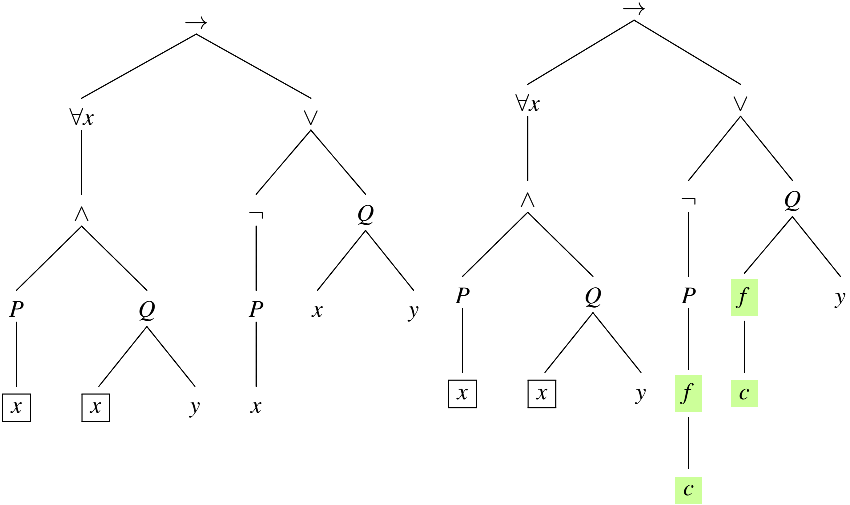

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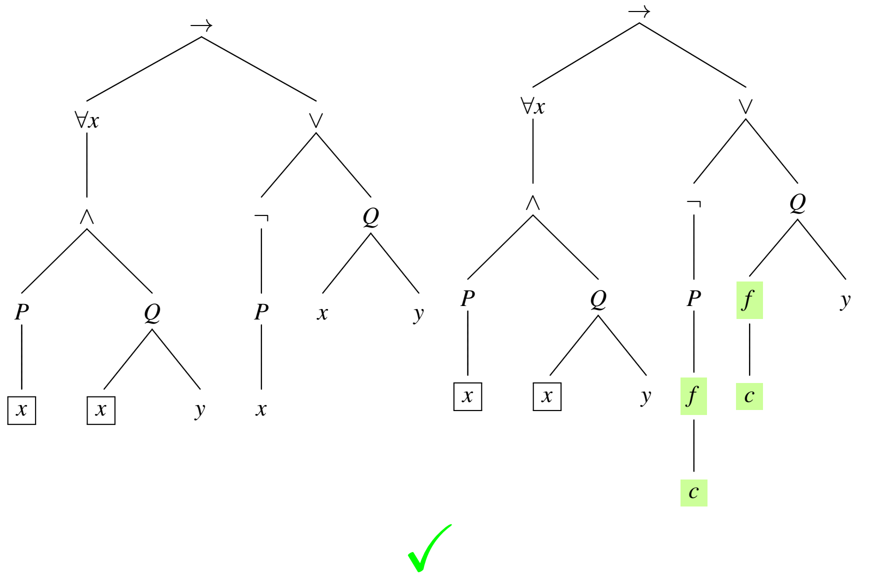

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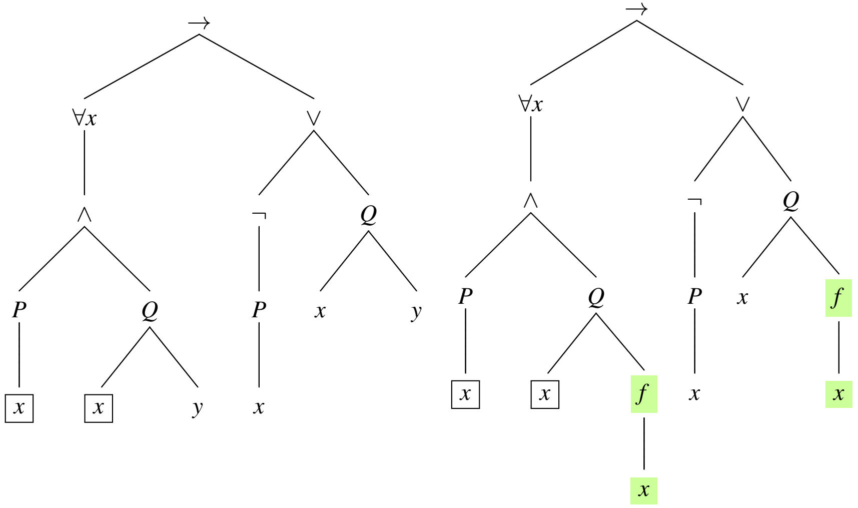

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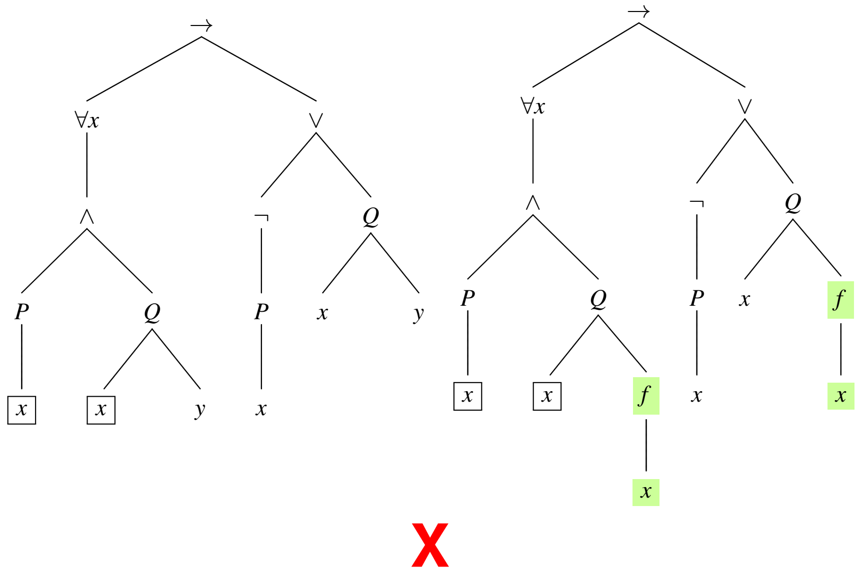

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\varphi[u/x] = \begin{cases} P(t_1[u/x], \ldots, t_n[u/x]) & \text{if } \varphi = P(t_1, \ldots, t_n) \\ (t_1[u/x] \doteq t_2[u/x]) & \text{if } \varphi = (t_2 \doteq t_2) \\ \neg(\varphi'[u/x]) & \text{if } \varphi = \neg\varphi' \\ \varphi_1[u/x] \star \varphi_2[u/x] & \text{if } \varphi = \varphi_1 \star \varphi_2 \text{ and} \\ & \star \in \{\wedge, \vee, \rightarrow\} \\ \mathbf{Q}y\,(\varphi'[u/x]) & \text{if } \varphi = \mathbf{Q}y\,\varphi', \\ & \mathbf{Q} \in \{\forall, \exists\},\ x \neq y, \text{ and} \\ & u \text{ is } \textbf{substitutable} \text{ for } x \text{ in } \varphi \\ \mathbf{Q}y\,\varphi' & \text{if } \varphi = \mathbf{Q}y\,\varphi', \\ & \mathbf{Q} \in \{\forall, \exists\},\ x = y \end{cases}
$$