CS 512, Spring 2017, Handout 16

Predicate Logic: Proof Rules of Natural Deduction

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proof rules for equality

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- equality introduction

$$\frac{}{t \doteq t} \doteq\mathrm{i}$$

proof rules for equality

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$$\frac{}{t \doteq t} \doteq\mathrm{i}$$

- equality elimination

$$\frac{t_1 \doteq t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]} \doteq\mathrm{e}$$

formal proof: "$\doteq$" is symmetric

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2	$u_1 \doteq u_1$	$\doteq$i
3	$u_2 \doteq u_1$	$\doteq$e $1, 2$

formal proof: "$\doteq$" is symmetric

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Answer: "$x \doteq u_1$" corresponds to φ in the rule $\doteq$e, so that "$u_1 \doteq u_1$" corresponds to $\varphi[u_1/x]$ & "$u_2 \doteq u_1$" corresponds to $\varphi[u_2/x]$

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$u_1 \doteq u_2 \ \vdash \ u_2 \doteq u_1$

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$u_1 \doteq u_2 \ \vdash \ u_2 \doteq u_1$

We can therefore use as a **derived proof rule**

$$\frac{t_1 \doteq t_2}{t_2 \doteq t_1} \quad \doteq \text{symmetric}$$

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formal proof: "$\doteq$" is transitive

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formal proof: "$\doteq$" is transitive

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$u_1 \doteq u_2,\ u_2 \doteq u_3 \ \vdash\ u_1 \doteq u_3$

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proof rules for universal quantification

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- universal quantifier elimination

$$\frac{\forall x\ \varphi}{\varphi[t/x]}\ \forall x\ \text{e}$$

(usual assumption: t is substitutable for x)

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$$\frac{\boxed{\begin{array}{ll} x_0 & \text{fresh} \\ \vdots & \\ \varphi[x_0/x] & \end{array}}}{\forall x\ \varphi}\ \forall x \text{ i}$$

proof rules for existential quantification

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- existential quantifier introduction

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$$\frac{\varphi[t/x]}{\exists x\ \varphi}\ \exists x\ \mathrm{i}$$

- existential quantifier elimination

$$\frac{\exists x\ \varphi \qquad \boxed{\begin{array}{lll} x_0 & & \text{fresh} \\ & \varphi[x_0/x] & \text{assumption} \\ & \vdots & \\ & \chi & \end{array}}}{\chi}\ \exists x\ \mathrm{e}$$

(x_0 cannot occur outside its box, in particular, it cannot occur in χ)

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- existential quantifier introduction

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(x_0 cannot occur outside its box, in particular, it cannot occur in χ)

- **Note carefully**:
  Rule $(\exists x\ \mathrm{e})$ introduces both a **fresh** variable and an **assumption**.

example: $\forall x\, \forall y\, \varphi(x,y) \;\vdash\; \forall y\, \forall x\, \varphi(x,y)$

	1	$\forall x\, \forall y\, \varphi(x,y)$	premise
y_0	2		fresh y_0
x_0	3		fresh x_0
	4	$\forall y\, \varphi(x_0,y)$	$\forall x$ e, 1
	5	$\varphi(x_0,y_0)$	$\forall x$ e, 4
	6	$\forall x\, \varphi(x,y_0)$	$\forall x$ i, 5
	7	$\forall y\, \forall x\, \varphi(x,y)$	$\forall y$ i, 6

example: $\forall x\,(P(x) \to Q(x)),\ \forall x\,P(x) \vdash \forall x\,Q(x)$

	1	$\forall x\,(P(x) \to Q(x))$	premise
	2	$\forall x\,P(x)$	premise
x_0	3		fresh x_0
	4	$P(x_0) \to Q(x_0)$	$\forall x$ e, 1
	5	$P(x_0)$	$\forall x$ e, 2
	6	$Q(x_0)$	$\to$e, 4, 5
	7	$\forall x\,Q(x)$	$\forall x$ i, 3-6

example: $\exists x\,(\varphi(x) \vee \psi(x)) \;\vdash\; \exists x\,\varphi(x) \;\vee\; \exists x\,\psi(x)$

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	3	$\varphi(x_0) \vee \psi(x_0)$		assumption
	4	$\varphi(x_0)$	$\psi(x_0)$	assumption
	5	$\exists x\,\varphi(x)$	$\exists x\,\psi(x)$	$\exists x$ i, 4
	6	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$	$\vee$i, 5
	7	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$		$\vee$e, 3, 4-6
	8	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$		$\exists x$ e, 1, 2-7

example: $\exists x\, \varphi(x) \;\vee\; \exists x\, \psi(x) \;\vdash\; \exists x\, (\varphi(x) \vee \psi(x))$

example: $\exists x\ \varphi(x)\ \lor\ \exists x\ \psi(x)\ \vdash\ \exists x\ (\varphi(x) \lor \psi(x))$

- Yes, this is a derivable sequent – left to you.

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- Hence, $\exists x\, \varphi(x) \;\vee\; \exists x\, \psi(x) \;\dashv\vdash\; \exists x\, (\varphi(x) \vee \psi(x))$

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  Find an interpretation (a "model") where

  $\exists x\,\varphi(x) \;\wedge\; \exists x\,\psi(x)$ is **true**, but

  $\exists x\,(\varphi(x) \wedge \psi(x))$ is **false**

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REMEMBER! To show that a WFF is **NOT** derivable, it is generally easier to find an interpretation where the WFF is not satisfiable.

example: $\exists x\, P(x), \forall x\, \forall y\, (P(x) \to Q(y)) \;\vdash\; \forall y\, Q(y)$

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	5	$P(x_0)$	assumption
	6	$\forall y\, (P(x_0) \to Q(y))$	$\forall x$ e, 2
	7	$P(x_0) \to Q(y_0)$	$\forall y$ e, 6
	8	$Q(y_0)$	$\to$e, 5, 7
	9	$Q(y_0)$	$\exists x$ e, 1, 4-8
	10	$\forall y\, Q(y)$	$\forall y$ i, 3-9

quantifier equivalences

Theorem

- $$\neg \forall x\, \varphi \quad \dashv\vdash \quad \exists x\, \neg\varphi$$
  $$\neg \exists x\, \varphi \quad \dashv\vdash \quad \forall x\, \neg\varphi$$

quantifier equivalences

Theorem

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- Assume x is not free in ψ:

quantifier equivalences

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- Assume x is not free in ψ:
  $$\forall x\,\varphi \wedge \psi \quad \dashv\vdash \quad \forall x\,(\varphi \wedge \psi)$$
  $$\forall x\,\varphi \vee \psi \quad \dashv\vdash \quad \forall x\,(\varphi \vee \psi)$$
  $$\exists x\,\varphi \wedge \psi \quad \dashv\vdash \quad \exists x\,(\varphi \wedge \psi)$$
  $$\exists x\,\varphi \vee \psi \quad \dashv\vdash \quad \exists x\,(\varphi \vee \psi)$$
  $$\forall x\,(\psi \to \varphi) \quad \dashv\vdash \quad \psi \to \forall x\,\varphi$$
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quantifier equivalences

Theorem

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- Assume x is not free in ψ:
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  $$\forall x\, (\psi \to \varphi) \quad \dashv\vdash \quad \psi \to \forall x\, \varphi$$
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  $$\exists x\, (\psi \to \varphi) \quad \dashv\vdash \quad \psi \to \exists x\, \varphi$$
- $$\forall x\, \varphi \wedge \forall x\, \psi \quad \dashv\vdash \quad \forall x\, (\varphi \wedge \psi)$$
  $$\exists x\, \varphi \vee \exists x\, \psi \quad \dashv\vdash \quad \exists x\, (\varphi \vee \psi)$$

proof of only one quantifier equivalence, others in the book

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	2	$\neg\exists x\,\neg\varphi$	assumption
x_0	3		fresh x_0
	4	$\neg\varphi[x_0/x]$	assumption
	5	$\exists x\,\neg\varphi$	$\exists x$ i, 4
	6	$\bot$	$\neg$e, 5, 2
	7	$\varphi[x_0/x]$	PBC, 4-6
	8	$\forall x\,\varphi$	$\forall x$ i, 3-7
	9	$\bot$	$\neg$e, 8, 1
	10	$\exists x\,\neg\varphi$	PBC, 2-9

three fundamental questions

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- **Question**
  Given a WFF φ, can we automate the answer to the query "$\vdash \varphi$??"

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  Given a WFF φ, can we automate the answer to the query "$\nvdash \varphi$??"
- **Question**
  Given a formal proof

  $$\begin{array}{ll} 1. & \varphi_1 \\ 2. & \varphi_2 \\ 3. & \vdots \\ n. & \varphi_n \end{array}$$

  can we automate the verification of the proof?