CS 512, Spring 2017, Handout 18

Examples of First-Order Theories

Assaf Kfoury

March 14, 2017

equivalence relations

1. $\forall x \quad (x \sim x)$ reflexivity
2. $\forall x\, \forall y \quad (x \sim y \rightarrow y \sim x)$ symmetry
3. $\forall x\, \forall y\, \forall z \quad (x \sim y \wedge y \sim z \rightarrow x \sim z)$ transitivity

equality with uninterpreted functions (EUF)

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2. $\forall x\, \forall y \quad (x \doteq y \to y \doteq x)$ symmetry
3. $\forall x\, \forall y\, \forall z \quad (x \doteq y \wedge y \doteq z \to x \doteq z)$ transitivity

The three preceding axioms are identical to those in the theory of **equivalence relations**.

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2. $\forall x\, \forall y \quad (x \doteq y \to y \doteq x)$ — symmetry
3. $\forall x\, \forall y\, \forall z \quad (x \doteq y \wedge y \doteq z \to x \doteq z)$ — transitivity

   The three preceding axioms are identical to those in the theory of **equivalence relations**.

4. for every function symbol $f \in \mathcal{F}$ of arity $n \geqslant 1$:

$$\forall x_1 \ \cdots \ \forall x_n \, \forall y_1 \ \cdots \ \forall y_n$$

$$\Big(\bigwedge_{1 \leqslant i \leqslant n} x_i \doteq y_i \Big) \to f(x_1, \ldots, x_n) \doteq f(y_1, \ldots, y_n) \qquad \text{congruence}$$

orders

1. $\forall x\, \forall y\, \forall z \quad \big(x \leqslant y \wedge y \leqslant z \to x \leqslant z\big)$ transitive
2. $\forall x \quad \big(x \leqslant x\big)$ reflexive
3. $\forall x\, \forall y \quad \big(x \leqslant y \wedge y \leqslant x \to x \doteq y\big)$ anti-symmetric

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4. $\forall x\, \forall y \quad \big(x \leqslant y \vee y \leqslant x\big)$ — total (or linear) ordering
5. $\forall x\, \forall z \quad \Big(x < z \rightarrow \exists y \quad \big(x < y \wedge y < z\big)\Big)$ — dense ordering

(where "$x < y$" abbreviates "$(x \leqslant y) \wedge \neg(x \doteq y)$")

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6. $\exists x\, \forall y \quad \big(x \leqslant y\big)$ — smallest element
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can we express a **well-ordering** in first-order logic? i.e., "every non-empty subset has a smallest element"?

algebras with two binary operations

1. $\forall x\, \forall y\, \forall z \quad \big(x \oplus (y \oplus z) \doteq (x \oplus y) \oplus z\big)$

   $\oplus$ is associative

2. $\forall x\, \forall y \quad \big(x \oplus y \doteq y \oplus x\big)$

   $\oplus$ is commutative

3. $\forall x\, \forall y\, \forall z \quad \big(x \otimes (y \oplus z) \doteq (x \otimes y) \oplus (x \otimes z)\big)$

   $\otimes$ distributes over $\oplus$

groups

1. $\forall x \quad \big(e \cdot x \doteq x \;\wedge\; x \cdot e \doteq x\big)$ identity (or neutral element)
2. $\forall x\, \exists y \quad \big(x \cdot y \doteq e \;\wedge\; y \cdot x \doteq e\big)$ inverse
3. $\forall x\, \forall y\, \forall z \quad \Big((x \cdot y) \cdot z \doteq x \cdot (y \cdot z)\Big)$ associative

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three preceding WFF's are true in every group,

does the following WFF follows from the preceding three:
$\forall x\, \forall y\, \forall z \quad \big(x \cdot y \doteq e \;\wedge\; x \cdot z \doteq e \rightarrow y \doteq z\big)$??

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some special cases of groups:

4. $\forall x\, \forall y \quad \big(x \cdot y \doteq y \cdot x\big)$ abelian group
5. $\forall x \quad \big(x \cdot x \doteq e \rightarrow x \doteq e\big)$, torsion-free group
   $\forall x \quad \big(x \cdot x \cdot x \doteq e \rightarrow x \doteq e\big)$,
   $\forall x \quad \big(x \cdot x \cdot x \cdot x \doteq e \rightarrow x \doteq e\big), \ldots$

graphs

1. $\forall x \, \forall y \quad \big(R(x,y) \;\rightarrow\; R(y,x)\big)$

   the graph is undirected

2. $\forall x \quad \big(\neg R(x,x)\big)$

   there are no “loops” in the graph

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assume there are two domains:
the domain V of vertices, the domain $\mathbb{R}$ of real numbers

assume there is a capacity function: $\quad c : V \times V \to \mathbb{R}$

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3. $\forall f\, \forall x\, \forall y \quad \big(f(x,y) \leqslant c(x,y)\big)$

   is (3) a first-order WFF?

successor function over the natural numbers

1. $\forall x \quad \big(\neg(\mathsf{S}\,x \doteq 0)\big)$
2. $\forall x\,\forall y \quad (\mathsf{S}\,x \doteq \mathsf{S}\,y \to x \doteq y)$
3. $\forall x \quad \big(\neg(x \doteq 0) \;\to\; \exists y\,(\mathsf{S}\,y \doteq x)\big)$

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4. for every WFF $\varphi(x)$ with a single free variable x, include the axiom
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   with **addition**:

5. $\forall x \quad (x + 0 \doteq x)$
6. $\forall x\, \forall y \quad \big(x + \mathsf{S}\,y \doteq \mathsf{S}\,(x + y)\big)$

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   with **addition**:

5. $\forall x \quad (x + 0 \doteq x)$
6. $\forall x\, \forall y \quad \big(x + \mathsf{S}\,y \doteq \mathsf{S}\,(x + y)\big)$

   with **addition** and **multiplication**:

7. $\forall x \quad (x \times 0 \doteq 0)$
8. $\forall x\, \forall y \quad \big(x \times \mathsf{S}\,y \doteq (x \times y) + x\big)$

linear integer arithmetic (LIA)

1. $\mathcal{P} = \{\leqslant\}, \quad \mathcal{F} = \{+, -\}, \quad \mathcal{C} = \{0, 1\}$.
2. include all axioms for "$+$" and "$-$".
3. atomic WFF's are all of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \bowtie b$$

where $\bowtie \in \{\leqslant, <, \geqslant, >, \doteq, \dot{\neq}\}$ and $a_1, \ldots, a_n, b$ are integers.