# CS 512, Spring 2017, Handout 23

# Extended Example in First-Order Logic

Assaf Kfoury

March 27, 2017

# several structures over the domain $\mathbb{N}$ (assume "$\doteq$" is available)

| structures over the domain of natural numbers | vocabulary/signature: predicate symbols | function symbols |
|---|---|---|
| $\mathcal{N} \triangleq (\mathbb{N}, 0, S)$ | $\mathscr{P} = \varnothing$ | $\mathscr{F} = \{0, S\}$ |
| $\mathcal{N}_1 \triangleq (\mathbb{N}, 0, S, <)$ | $\mathscr{P} = \{<\}$ | $\mathscr{F} = \{0, S\}$ |
| $\mathcal{N}_2 \triangleq (\mathbb{N}, 0, S, <, +)$ | $\mathscr{P} = \{<\}$ | $\mathscr{F} = \{0, S, +\}$ |
| $\mathcal{N}_3 \triangleq (\mathbb{N}, 0, S, <, +, \cdot)$ | $\mathscr{P} = \{<\}$ | $\mathscr{F} = \{0, S, +, \cdot\}$ |
| $\mathcal{N}_4 \triangleq (\mathbb{N}, 0, S, <, +, \cdot, \text{pr})$<br>$\text{pr}(x) \triangleq$ true iff $x$ is prime | $\mathscr{P} = \{<, \text{pr}\}$ | $\mathscr{F} = \{0, S, +, \cdot\}$ |
| $\mathcal{N}_5 \triangleq (\mathbb{N}, 0, S, <, +, \cdot, \text{pr}, \uparrow)$<br>$x \uparrow y \triangleq x^y$ | $\mathscr{P} = \{<, \text{pr}\}$ | $\mathscr{F} = \{0, S, +, \cdot, \uparrow\}$ |
| $\mathcal{N}_6 \triangleq \cdots$ | | |

Question: Is a new predicate (function) definable from earlier ones?

# first-order definability over $\mathbb{N}$

- every number $n$ is definable from $0$ and $S$:

$$
\begin{aligned}
1 &\triangleq S(0) \\
2 &\triangleq S(S(0)) \\
3 &\triangleq S(S(S(0))) \\
&\dots \\
n &\triangleq \underbrace{S(\cdots S}_{n}(0)\cdots)
\end{aligned}
$$

# first-order definability over $\mathbb{N}$

- every number $n$ is definable from $0$ and $S$:

$$
\begin{aligned}
1 &\triangleq S(0) \\
2 &\triangleq S(S(0)) \\
3 &\triangleq S(S(S(0))) \\
&\cdots \\
n &\triangleq \underbrace{S(\cdots S}_{n}(0)\cdots)
\end{aligned}
$$

- "$S$" is definable from "$+$":

  for all $m, n \in \mathbb{N}$, we have $S(m) = n$ iff $m + 1 = n$

# first-order definability over $\mathbb{N}$

- every number $n$ is definable from $0$ and $S$:

  $$\begin{aligned} 1 &\triangleq S(0) \\ 2 &\triangleq S(S(0)) \\ 3 &\triangleq S(S(S(0))) \\ &\cdots \\ n &\triangleq \underbrace{S(\cdots S}_{n}(0)\cdots) \end{aligned}$$

- "$S$" is definable from "$+$":

  for all $m, n \in \mathbb{N}$, we have $S(m) = n$ iff $m + 1 = n$

  formally: the sentence $\forall x \forall y\,(\ S(x) \doteq y \ \leftrightarrow\ x + 1 \doteq y\ )$ is true in $\mathcal{N}_2$,

  which implies the graph of $S^{\mathcal{N}_2}$ is defined by the WFF $(x + 1 \doteq y)$.

# first-order definability over $\mathbb{N}$

- every number $n$ is definable from $0$ and $S$:

  $$\begin{aligned} 1 &\triangleq S(0) \\ 2 &\triangleq S(S(0)) \\ 3 &\triangleq S(S(S(0))) \\ &\cdots \\ n &\triangleq \underbrace{S(\cdots S}_{n}(0)\cdots) \end{aligned}$$

- "$S$" is definable from "$+$":

  for all $m, n \in \mathbb{N}$, we have $S(m) = n$ iff $m + 1 = n$

  formally: the sentence $\forall x \forall y \, ( \, S(x) \doteq y \; \leftrightarrow \; x + 1 \doteq y \, )$ is true in $\mathcal{N}_2$,

  which implies the graph of $S^{\mathcal{N}_2}$ is defined by the WFF $(x + 1 \doteq y)$.

- is "$+$" definable from "$S$"? perhaps ...

# first-order definability over $\mathbb{N}$

- every number $n$ is definable from $0$ and $S$:

  $$\begin{aligned} 1 &\triangleq S(0) \\ 2 &\triangleq S(S(0)) \\ 3 &\triangleq S(S(S(0))) \\ &\cdots \\ n &\triangleq \underbrace{S(\cdots S}_{n}(0)\cdots) \end{aligned}$$

- "$S$" is definable from "$+$":

  for all $m, n \in \mathbb{N}$, we have $S(m) = n$ iff $m + 1 = n$

  formally: the sentence $\forall x \forall y \, (\; S(x) \doteq y \;\leftrightarrow\; x + 1 \doteq y \;)$ is true in $\mathcal{N}_2$,

  which implies the graph of $S^{\mathcal{N}_2}$ is defined by the WFF $(x + 1 \doteq y)$.

- is "$+$" definable from "$S$"? perhaps ...

  for all $m, n, p \in \mathbb{N}$, we have $m + n = p$ iff $\underbrace{S(\cdots S}_{n}(m)\cdots) = p$

# first-order definability over $\mathbb{N}$

- every number $n$ is definable from $0$ and $S$:

  $$1 \triangleq S(0)$$
  $$2 \triangleq S(S(0))$$
  $$3 \triangleq S(S(S(0)))$$
  $$\dots$$
  $$n \triangleq \underbrace{S(\cdots S}_{n}(0)\cdots)$$

- "$S$" is definable from "$+$":

  for all $m, n \in \mathbb{N}$, we have $S(m) = n$ iff $m + 1 = n$

  formally: the sentence $\forall x \forall y\,(\ S(x) \doteq y \ \leftrightarrow\ x + 1 \doteq y\ )$ is true in $\mathcal{N}_2$,

  which implies the graph of $S^{\mathcal{N}_2}$ is defined by the WFF $(x + 1 \doteq y)$.

- is "$+$" definable from "$S$"? perhaps ...

  for all $m, n, p \in \mathbb{N}$, we have $m + n = p$ iff $\underbrace{S(\cdots S}_{n}(m)\cdots) = p$

  "formally": $\forall x \forall y \forall z\,[\ \underbrace{S(\cdots S}_{y}(x)\cdots) \doteq z \ \leftrightarrow\ x + y \doteq z\ ]$

# first-order definability over $\mathbb{N}$

1. **FACT**

   "$+$" is **NOT** (first-order) definable from "$0$" and "$S$" *(difficult!)*

# first-order definability over $\mathbb{N}$

1. **FACT**

   "$+$" is **NOT** (first-order) definable from "$0$" and "$S$" *(difficult!)*

2. **FACT**

   "$<$" is (first-order) definable from "$+$" *(easy: try it!)*

# first-order definability over $\mathbb{N}$

1. **FACT**
   "$+$" is **NOT** (first-order) definable from "$0$" and "$S$" *(difficult!)*
2. **FACT**
   "$<$" is (first-order) definable from "$+$" *(easy: try it!)*
3. **FACT**
   "$+$" is **NOT** (first-order) definable from "$<$", "$0$", and "$S$" *(difficult!)*

# first-order definability over $\mathbb{N}$

1. **FACT**
   "$+$" is **NOT** (first-order) definable from "$0$" and "$S$" *(difficult!)*
2. **FACT**
   "$<$" is (first-order) definable from "$+$" *(easy: try it!)*
3. **FACT**
   "$+$" is **NOT** (first-order) definable from "$<$", "$0$", and "$S$" *(difficult!)*
4. **FACT**
   "$\cdot$" is **NOT** (first-order) definable from "$0$", "$S$", and "$+$"
   (no need to mention "$<$") *(difficult!)*

# first-order definability over $\mathbb{N}$

1. **FACT**
   "$+$" is **NOT** (first-order) definable from "$0$" and "$S$" *(difficult!)*
2. **FACT**
   "$<$" is (first-order) definable from "$+$" *(easy: try it!)*
3. **FACT**
   "$+$" is **NOT** (first-order) definable from "$<$", "$0$", and "$S$" *(difficult!)*
4. **FACT**
   "$\cdot$" is **NOT** (first-order) definable from "$0$", "$S$", and "$+$"
   (no need to mention "$<$") *(difficult!)*
5. **FACT**
   "$+$" is (first-order) definable from "$<$" and "$\cdot$" *(tricky: try hint below!)*

# first-order definability over $\mathbb{N}$

1. **FACT**
   "$+$" is **NOT** (first-order) definable from "$0$" and "$S$" *(difficult!)*
2. **FACT**
   "$<$" is (first-order) definable from "$+$" *(easy: try it!)*
3. **FACT**
   "$+$" is **NOT** (first-order) definable from "$<$", "$0$", and "$S$" *(difficult!)*
4. **FACT**
   "$\cdot$" is **NOT** (first-order) definable from "$0$", "$S$", and "$+$"
   (no need to mention "$<$") *(difficult!)*
5. **FACT**
   "$+$" is (first-order) definable from "$<$" and "$\cdot$" *(tricky: try hint below!)*

*Hint*. Use the following equivalence for all $m, n, p \in \mathbb{N}$
$(p = 0) \vee (p = m + n)$ iff
$(m \cdot p + 1) \cdot (n \cdot p + 1) = p^2 \cdot (m \cdot n + 1) + 1$

# first-order definability over $\mathbb{N}$

- is “pr” definable from $\{0, S, <, +, \cdot\}$?

# first-order definability over $\mathbb{N}$

- is "pr" definable from $\{0, S, <, +, \cdot\}$?

  **YES** pr$(n)$ is true iff $\varphi(n)$ is true, where $\varphi(x)$ is the WFF

  $$\varphi(x) \;\triangleq\; \neg(x \doteq 1) \;\wedge\; \forall y \forall z\, [\, (x \doteq y \cdot z) \;\rightarrow\; (y \doteq 1 \vee z \doteq 1)\, ]$$

- is "$\uparrow$" definable from $\{0, S, <, +, \cdot\}$?

# first-order definability over $\mathbb{N}$

- is "pr" definable from $\{0, S, <, +, \cdot\}$?

  **YES** pr$(n)$ is true iff $\varphi(n)$ is true, where $\varphi(x)$ is the WFF

  $\varphi(x) \;\triangleq\; \neg(x \doteq 1) \;\wedge\; \forall y \forall z\,[\,(x \doteq y \cdot z) \;\rightarrow\; (y \doteq 1 \vee z \doteq 1)\,]$

- is "$\uparrow$" definable from $\{0, S, <, +, \cdot\}$?

  **YES** $m = n \uparrow p$ iff $\varphi(m, n, p)$ is true, where $\varphi(x, y, z)$ is the WFF ... *(not very difficult: try it!)*