CS 512, Spring 2017, Handout 25

Gilmore's Algorithm

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review and reminders *(run simultaneously with an example on the board)*

From the handout *Compactness+Completeness* (click here to retrieve it):

- If φ is a first-order sentence, then $\boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)$ is its Skolem form.
- In particular, $\boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)$ is a universal first-order sentence, *i.e.*, it is in prenex normal form and all the quantifiers in its prenex are universal.
- φ and $\boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)$ are equisatisfiable
(Lemma 21 in *Compactness+Completeness*).

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- Gr_Expansion$\left(\Theta_{\text{pr,sk}}(\varphi)\right)$ is obtained by deleting the prenex of $\Theta_{\text{pr,sk}}(\varphi)$ and substituting ground terms for variables in the matrix of $\Theta_{\text{pr,sk}}(\varphi)$.
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- φ and Gr_Expansion$\big(\boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)\big)$ are equisatisfiable (Lemma 28 in *Compactness+Completeness*).
- $\mathcal{X}\Big($Gr_Expansion$\big(\boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)\big)\Big)$ is obtained by replacing every ground atom α in Gr_Expansion$\big(\boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)\big)$ by a propositional variable X_α.
- φ is satisfiable (in first-order logic) iff $\mathcal{X}\Big($Gr_Expansion$\big(\boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)\big)\Big)$ is satisfiable (in propositional logic). (Theorem 32 in *Compactness+Completeness*).

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- φ is satisfiable (in first-order logic) iff
  $\mathcal{X}\big(\text{Gr\_Expansion}\big(\boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)\big)\big)$ is **finitely** satisfiable (in prop logic).

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  φ is **not** satisfiable (in first-order logic) iff
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- Recall that a first-order sentence ψ is **valid** iff $\neg\psi$ is **not** satisfiable.

  Suppose we want to test whether a first-order sentence ψ is valid. Let

  $$\mathcal{X}\big(\text{Gr\_Expansion}\big(\boldsymbol{\Theta}_{\text{pr,sk}}(\neg\psi)\big)\big) = \{\theta_1,\ \theta_2,\ \theta_3, \ldots\}$$

  Note the inserted logical negation "$\neg$". All the θ_i's are propositional WFF's.

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Gilmore's algorithm

1. **input**: first-order sentence ψ to be tested for validity;
2. $k := 0$;
3. **repeat** $k := k + 1$

   until $\bigwedge_{1 \leqslant i \leqslant k} \theta_i$ is unsatisfiable;
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Exercise: Let $\varphi_1, \ldots, \varphi_n$ and ψ be first-order sentences.
Define an algorithm based on Gilmore's algorithm which terminates iff the semantic entailment $\varphi_1, \ldots, \varphi_n \models \psi$ holds.

Problem: Can you define an algorithm $\mathcal{A}$ which, given a first-order sentence ψ, always terminates and decides whether ψ is valid or not valid?
Hint: No.

Gilmore's algorithm

- Gilmore's algorithm is said to be a semi-decision procedure, because it terminates only if the input ψ is valid.
- Gilmore's algorithm was invented in the late 1950's and it was the best semi-decision procedure for first-order validity until the mid-1960's, when more efficient early versions of the tableaux and resolution methods were first introduced.