# CS 512, Spring 2017, Handout 32

# Second Order Logic

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# example

- Let $\varphi \triangleq \exists y\,[\,P(y) \to \forall x\,P(x)\,]$

  $\varphi$ is a first-order sentence over the vocabulary/signature $\Sigma = \{P\}$.

  Is $\varphi$ semantically valid (true in every model) or, equivalently, formally provable?

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- Yes, it is, no matter the interpretation of the predicate symbol $P$.

  So why not consider instead the formula $\psi \triangleq \forall P \, \varphi$?

  **$\psi$ is no longer first-order, but a second-order sentence.**

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- Do we have a formal semantics for second-order logic?

  Do we have a formal proof theory / deductive system for second-order logic?

  If the answer is **yes** to both questions, do we a have soundness-and-completeness theorem for second-order logic?

# from first-order to second-order logic

Given a vocabulary $\Sigma = \mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ as before –

$\mathcal{P}$ is a collection of predicate symbols,
$\mathcal{F}$ a collection of function symbols,
$\mathcal{C}$ a collection of constant symbols –

we go from the syntax and formation rules of first-order logic to second-order logic by adding:

- **predicate variables**: $X_1, X_2, \ldots$ each with a fixed arity $n \geqslant 1$.
- **function variables**: $F_1, F_2, \ldots$ each with a fixed arity $n \geqslant 1$.

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The definition of a model $\mathcal{M}$ proceeds as in Handout 17, except that now an **environment** (or **look-up table**) $\ell$ must assign a meaning to **predicate variables** and **function variables**, in addition to **individual variables**.

# from first-order to second-order logic

The only new features in the definition of ***satisfaction*** deal with the second-order quantifiers – see Handout 17:

- let $X$ be a $n$-ary predicate variable, for some $n \geqslant 1$,

$$\mathcal{M}, \ell \models \forall X\, \varphi \quad \text{iff } \mathcal{M}, \ell[X \mapsto R] \models \varphi \text{ for every } R \subseteq \underbrace{A \times \cdots \times A}_{n}$$

- let $F$ be a $n$-ary function variable, for some $n \geqslant 1$,

$$\mathcal{M}, \ell \models \forall F\, \varphi \quad \text{iff } \mathcal{M}, \ell[F \mapsto f] \models \varphi \text{ for every } f : \underbrace{A \times \cdots \times A}_{n} \to A$$

# semantic entailment, semantic validity, satisfiability

Let $\varphi$ be a second-order WFF. Similar to 1st order logic, we say:

- WFF $\varphi$ is **satisfiable** iff there are some $\mathcal{M}$ and $\ell$ such that $\mathcal{M}, \ell \models \varphi$
- WFF $\varphi$ is **semantically valid** iff for all $\mathcal{M}$ and $\ell$ it holds that $\mathcal{M}, \ell \models \varphi$
- If $\varphi$ is a closed second-order WFF, we write $\mathcal{M} \models \varphi$ instead of $\mathcal{M}, \ell \models \varphi$

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Let $\Gamma$ be a set of second-order WFF's:

- $\Gamma$ is **satisfiable** iff there are some $\mathcal{M}$ and $\ell$ s.t. $\mathcal{M}, \ell \models \varphi$ for every $\varphi \in \Gamma$
- **semantic entailment**: $\Gamma \models \psi$ iff for every $\mathcal{M}$ and every $\ell$, it holds that $\mathcal{M}, \ell \models \Gamma$ implies $\mathcal{M}, \ell \models \psi$

# soundness and completeness for second-order logic ???

- There are several deductive systems for second-order logic, but none can be **complete** w.r.t. second-order semantics.
  (Not shown in this handout.)
- At a minimum, each of these deductive systems is **sound**, i.e., any second-order WFF which is formally derivable is semantically valid.
  (Not shown in this handout.)

# examples (modeling in second-order logic)

- "A **well-ordering** is an ordering $\leqslant$ such that every non-empty set has a least element w.r.t. $\leqslant$"
- From Handout 18: Can first-order logic specific a well-ordering?

# examples (modeling in second-order logic)

- "A **well-ordering** is an ordering $\leqslant$ such that every non-empty set has a least element w.r.t. $\leqslant$"
- From Handout 18: Can first-order logic specific a well-ordering?
- Second-order logic can express the well-ordering property:

$$\varphi \triangleq \forall X \left( \exists y\, X(y) \to \exists v \left(X(v) \wedge \forall w\, (X(w) \to v \leqslant w)\right) \right)$$

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- "A **well-ordering** is an ordering $\leqslant$ such that every non-empty set has a least element w.r.t. $\leqslant$"
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- Second-order logic can express the well-ordering property:

$$\varphi \triangleq \forall X \left( \exists y\, X(y) \to \exists v\, \big(X(v) \land \forall w\, (X(w) \to v \leqslant w)\big) \right)$$

- **Fact** (not proved here): The set of sentences

  $$\{\varphi\} \cup \mathsf{Th}(\mathcal{N}_1)$$

  defines $\mathcal{N}_1$ (and every structure which is an expansion of $\mathcal{N}_1$) **up to isomorphism**, where $\mathcal{N}_1 \triangleq (\mathbb{N}, 0, S, <)$ in Handout 23.
- **Fact** (not proved here): First-order logic cannot specify the well-ordering property, because there are non-isomorphic models of $\mathsf{Th}(\mathcal{N}_1)$, some of which are well-ordered and some are not well-ordered.

# examples (modeling in second-order logic)

- A second-order sentence satisfied by a structure $\mathcal{M}$ iff the domain/universe of $\mathcal{M}$ is **infinite**:[1]

$$\begin{aligned}
\psi \triangleq \exists P \Big( & \forall x\, \forall y\, \forall z\, \big(P(x,y) \wedge P(y,z) \rightarrow P(x,z)\big) && \text{“$P$ is transitive”} \\
\wedge \quad & \forall x\, \big(\neg P(x,x)\big) && \text{“$P$ is not reflexive”} \\
\wedge \quad & \forall x\, \exists y\, P(x,y) \Big) && \text{“every $x$ is s.t. $x \xrightarrow{P} y$ for some $y$”}
\end{aligned}$$

[1] By definition, the universe of $\mathcal{M}$, is a non-empty set. Hence, $\psi$ cannot be vacuously true, because all models of $\psi$ have non-empty universes.

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$$\wedge \quad \forall x \, \exists y \, P(x,y) \Big) \qquad \text{“every } x \text{ is s.t. } x \xrightarrow{P} y \text{ for some } y\text{”}$$

- A second-order sentence satisfied by a model $\mathcal{M}$ iff the domain of $\mathcal{M}$ is **finite**:

$\neg\psi$

[1] By definition, the universe of $\mathcal{M}$, is a non-empty set. Hence, $\psi$ cannot be vacuously true, because all models of $\psi$ have non-empty universes.

## compactness and completeness fail for second-order logic

**Compactness Theorem for First-Order**

Let $\Gamma$ be a set of first-order sentences.

1. If every finite subset of $\Gamma$ is **satisfiable**, then so is $\Gamma$.
2. If every finite subset of $\Gamma$ is **consistent**, then so is $\Gamma$.

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**Counter-Example for Second-Order Compactness**

For every $n \geqslant 1$, define the first-order sentence $\theta_n$ by:

$$\theta_n \triangleq \quad \text{“there are at least } n \text{ distinct elements”}$$

Consider the set of sentences:

$$\Delta = \{\neg\psi\} \cup \{\theta_1, \theta_2, \theta_3, \ldots\}$$

Every finite subset of $\Delta$ is **satisfiable**, while $\Delta$ is **unsatisfiable**.

# compactness and completeness fail for second-order logic

- There are deductive systems (*i.e.*, formal proof theories) for second-order logic, but none can be complete (for the standard semantics).

  In contrast to first-order logic:
  "There are deductive systems for first-order logic which are complete."

- There are sets $\Gamma$ of second-order sentences which, although consistent (*i.e.*, $\bot$ cannot be formally deduced from $\Gamma$), do not have models.

  In contrast to first-order logic:
  "Every consistent set of first-order sentences has a model."