CS 512, Spring 2017, Handout 33

Second Order Logic
(with several examples in formal modeling)

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examples with graphs (A, R)

where A is the set of nodes and R is a binary relation representing edges

- "A **Hamiltonian path** is a path that visits every node exactly once"

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$$\varphi \triangleq \exists P \Big(\; \psi_1(P) \; \wedge \; \forall x \forall y \, (\; \psi_2(P, x, y) \; \rightarrow R(x, y)) \Big)$$

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$\psi_1(P)$ makes predicate-variable P a linear order:

$\psi_1(P) \triangleq\ \forall x\, P(x, x)\ \wedge$	reflexivity
$\forall x \forall y \forall z\, \big(P(x, y) \wedge P(y, z) \rightarrow P(x, z)\big)\ \wedge$	transitivity
$\forall x \forall y\, \big(P(x, y) \wedge P(y, x) \rightarrow x \doteq y\big)\ \wedge$	anti-symmetry
$\forall x \forall y\, \big(P(x, y) \vee P(y, x)\big)$	totality

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$\forall x \forall y\, \big(P(x, y) \lor P(y, x)\big)$	totality

$\psi_2(P, x, y)$ is a WFF with free predicate-variable P of arity 2 and first-order variables x and y, which makes y the successor of x in the linear order P:

$$\psi_2(P, x, y) \triangleq \neg(x \doteq y) \land P(x, y) \land \forall z\, \big(P(x, z) \land P(z, y) \to (x \doteq z \lor y \doteq z)\big)$$

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  $$\varphi \triangleq \exists P \forall x \forall y \Big(\neg(x \doteq y) \wedge R(x,y) \to (P(x) \leftrightarrow \neg P(y)) \Big)$$

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  represent 3 colors by unary predicate variables A_1, A_2, and A_3
- ψ_1 says "each node has exactly one color":

$$\psi_1(A_1, A_2, A_3) \triangleq \forall x \Big(\big(A_1(x) \wedge \neg A_2(x) \wedge \neg A_3(x) \big) \vee \big(\neg A_1(x) \wedge A_2(x) \wedge \neg A_3(x) \big) \vee \big(\neg A_1(x) \wedge \neg A_2(x) \wedge A_3(x) \big) \Big)$$

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- ψ_2 says "no two points with the same color are connected":

$$\psi_2(A_1, A_2, A_3) \triangleq \forall x \forall y \Big(\big(A_1(x) \wedge A_1(y) \to \neg R(x, y) \big) \wedge \big(A_2(x) \wedge A_2(y) \to \neg R(x, y) \big) \wedge \big(A_3(x) \wedge A_3(y) \to \neg R(x, y) \big) \Big)$$

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- $\varphi \triangleq \exists A_1 \exists A_2 \exists A_3 \, (\psi_1 \wedge \psi_2)$

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- ψ_1 says "the set A is non-empty and its complement is nonempty"

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$$\psi_2(A) \triangleq \forall x \forall y \; \Big(\; \big(A(x) \wedge \neg A(y)\big) \to \big(\neg R(x,y) \wedge \neg R(y,x)\big) \; \Big)$$

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  is true iff graph **is not connected**

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- $\varphi \ \triangleq\ \exists A\,(\psi_1 \wedge \psi_2)$
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- $\varphi' \ \triangleq\ \neg\varphi \ \triangleq\ \forall A\,(\neg\psi_1 \vee \neg\psi_2) \ \triangleq\ \forall A\,(\psi_1 \rightarrow \neg\psi_2)$
  is true iff graph **is connected**

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- **reachability**

  Example 2.27 in [LCS. page 140].

connections with *descriptive complexity theory*

- Starting point:

  Syntactic classification of second-order WFF's in prenex normal form, over a given signature Σ, according to:

  1. interleaving of universal and existential quantifiers in the prenex, and
  2. arities of predicate and function symbols in Σ.

- **Example**:
  The WFF φ in each on slide 4, slide 7, slide 11, and slide 16 is an existential second-order WFF.

- **Example**:
  The φ in each of slide 7, slide 11, and slide 16, but not on slide 4, is a monadic second-order WFF, because the second-order variables in φ are restricted to be unary-predicate (*i.e.*, set) variables.

- **Example**:
  Monadic second-order logic has been extensively studied in relation to graph properties and their complexities. (Search the WWW with the keyword "monadic second-order logic.")

connections with *descriptive complexity theory*

- Prototypical result of descriptive complexity theory:

  Fagin's theorem: Let $\mathcal{C}$ be the class of all finite undirected graphs (closed under isomorphism). The following are equivalent statements:

  1. $\mathcal{C}$ is in NP.
  2. $\mathcal{C}$ is definable by an existential second-order sentence.

  In fact, every class of objects in NP has an existential second-order characterization with binary predicates and a universal first-order part.