CS 512 Formal Methods, Spring 2017	Instructor: Assaf Kfoury
Lecture 1	
January 19th, 2017	Sean Smith

(These lecture notes are **not** proofread and proof-checked by the instructor.)

General Notes:

- If you volunteer to scribe a lecture you'll get credit
- If you grade a homework assignment you'll get full credit for that assignment
- This semester with focus on SMT/SAT solvers

Each Well Formed Formula (WFF) is constructed by applying one of the below construction rules finitely many times:

1. Every propositional variable p is a WFF.
2. If φ is a WFF then so is $\neg\varphi$
3. If φ and ψ are WFF's then so is $\varphi \wedge \psi$
4. If φ and ψ are WFF's then so is $\varphi \vee \psi$
5. If φ and ψ are WFF's then so is $\varphi \to \psi$

This can succinctly be written in Backus Naur Form (BNF) like so:

$$\varphi ::= p|(\neg\varphi)|(\varphi \wedge \psi)|(\varphi \vee \psi)|(\varphi \to \psi)$$

We can write it in extended BNF without parenthesis and using the same variable for both sides (note that even though the only symbol I'm using is φ, it does not mean that both sides of the equation are true).

$$\varphi ::= p|\neg\varphi|\varphi \wedge \varphi|\varphi \vee \varphi|\varphi \to \varphi$$

If you're writing the equation as a parse tree there's not need for parenthesis and no need to disambiguate proper ordering. However since it's far easier write horizontally, use parenthesis, but only sparingly.

Natural Deduction Example:
Let's consider the sentence:

If the train arrives late and there are no taxis then John is late to the meeting.

Let's assign variables to each part of the above sentence so we can write it as a sequent.

1. P train arrives late

2. Q there are taxis
3. R John is late for the meeting

Then we can write:

$$P \land \neg Q \vdash R$$

Let's say we know P and $\neg R$ are true. We can write a proof that Q is true like so:

1	$(P \land \neg Q) \to R$	premise
2	$\neg R$	premise
3	P	premise
4	$\neg Q$	assumption
5	$P \land \neg Q$	$\land$i $3, 4$
6	R	$\neg e 1, 5$
7	$\bot$	$\neg e 6, 2$
8	$\neg\neg Q$	$\neg$i
9	Q	$\neg\neg e 8$

This proof is derived using proof rules which are used to prove a sequent. A sequent is an expression of the form $\varphi_1, \ldots, \varphi_n \vdash \psi$ where each φ_i and ψ is a WFF.

The proof rules are summarized below. Please note that i means introduction and e means elimination. So when read a statement like $\land i$ 3, interpret it as introduction of $\land$ to line 3. the top of the fraction is the premise and the bottom is the conclusion, so $\frac{\varphi \quad \psi}{\varphi \land \psi}$ can be read as, if φ is true and ψ is true then $\varphi \land \psi$ is true. Pretty simple right?

1. $\frac{\varphi \quad \psi}{\varphi \land \psi} \land i$
2. $\frac{\varphi \land \psi}{\varphi} \land e_1$
3. $\frac{\varphi \land \psi}{\psi} \land e_2$
4. $\frac{\varphi}{\neg\neg\varphi} \neg\neg i$
5. $\frac{\neg\neg\varphi}{\varphi} \neg\neg e$
6. $\frac{\varphi \quad \varphi \to \psi}{\psi} \to e$ This is also referred to as Modus Ponens (MP)
7. $\frac{\varphi \to \psi \quad \neg\psi}{\neg\varphi}$ Modus Tollens (MT) or contrapositive
8. $\dfrac{\boxed{\begin{array}{ll} 1 & \varphi \\ 2 & \ldots \\ 3 & \psi \end{array}}}{\varphi \to \psi} \to i$

- Open a box when you introduce an assumption
- Close the box when you terminate an assumption

9. $\frac{\varphi \quad \neg\varphi}{\bot}$ $\neg e$ Law of non-contradiction

10. $\dfrac{\boxed{\begin{array}{ll} 1 & \varphi \\ 2 & \dots \\ 3 & \bot \end{array}}}{\neg\bot}$ $\neg i$

11. $\frac{\bot}{\varphi}$ $\bot e$ If you can prove falsity then you can prove anything.

12. $\dfrac{\boxed{\begin{array}{ll} 1 & \neg\varphi \\ 2 & \dots \\ 3 & \bot \end{array}}}{\varphi}$ Proof by contradiction. (PBC)

13. $\frac{}{\varphi \vee \neg\varphi}$ Law of excluded middle