CS 512 Formal Methods, Spring 2017	Instructor: Assaf Kfoury

Lecture 3: Propositional Logic - Soundness, Completeness and Compactness

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(These lecture notes are **not** proofread and proof-checked by the instructor.)

General Information:

- Lecture notes (by scribes) are not proof-checked by the instructor. They are a service from the from the scribe to the rest of class.
- One page project proposal due on Friday, 02/03.
- One dropbox folder to submit each assignment.
- Open area of research: theory + implementation of SAT solvers for inutitionistic propositional logic.

Curry-Howard Correspondence

Programs and intuitionistic logic have a very high correspondence between them. Programs - as proofs, Types- as formulas.

Soundness

Let Γ be the knowledge base. If $\Gamma \vdash \psi$ is valid, then $\Gamma \models \psi$ holds.

Completeness

$\Gamma \vdash \psi$ is valid iff $\Gamma \models \psi$ holds. If Γ is infinite, we need another preliminary result - compactness.

Suppose ψ is a WFF with 100 propositional variables/atoms.

Q1. Can you decide if ψ is satisfiable? Yes.

Q2. If yes, can you produce a formal proof for ψ? Yes.

Q3. Can you systematically generate all WFFs of a propositional logic? Yes.

Example: Given three propositional atoms x_1, x_2, x_3 and connectives $\neg$ and $\wedge$ then

WFFs of size 1: x_1, x_2, x_3

WFFs of size 2: $\neg x_1$, $\neg x_2$, $\neg x_3$

WFFs of size 3: $x_1 \wedge x_2$, $x_2 \wedge x_3$, $x_1 \wedge x_3$

Dove-tailing:

1. Produce a formal proof of length 1.

2. Produce a formal proof of length 2.

3. Then use dove-tailing method to find all WFFs.

Q4. Can you systematically generate all well-formed natural deduction formal proofs? Yes.

Lemma: Suppose Γ is finitely satisfiable. Then for every WFF ψ, either $\Gamma \cup \{\psi\}$ or $\Gamma \cup \{\neg\psi\}$ is finitely satisfiable.

Proof:

Let X = $\{x_1, x_2, x_3......\}$

Let ψ_1, ψ_2, ψ_3 ... be a fixed enumeration of all the WFFs of propositional logic over X. X $\cup$ { $\wedge$, $\neg$ }

Define a nested sequence of supersets of Γ as: $\triangle_0 \subset \triangle_1 \subset \triangle_3...$

$$\triangle_0 = \Gamma$$

$$\triangle_{i+1} = \begin{cases} \triangle_i \cup \{\psi_i\}, & \text{if } \triangle_i \cup \{\psi_i\} \text{ is finitely satisfiable.} \\ \triangle_i \cup \{\neg\psi\}, & \text{otherwise.} \end{cases} \quad (1)$$

Define a set $\triangle = \bigcup \triangle_i$

Facts:

(a) For every WFF ψ of propositional logic, either $\psi \in \triangle$ or $\psi \notin \triangle$.

(b) For every proposition x_i either $x_i \in \triangle$ or $\neg x_i \in \triangle$.

Define a boolean evaluation of σ as follows:

$$\sigma(x_i) = \begin{cases} True, & \text{if } x_i \in \triangle \\ False, & \text{if } x_i \notin \triangle \end{cases} \quad (2)$$

Claim:

σ satisfies $\triangle$

1. Take an arbitrary ψ in $\triangle$.

2. Prove by structural induction that σ satisfies ψ.

Hence, σ satisfies Γ.