CS 512 Formal Methods, Spring 2017	Instructor: Assaf Kfoury

Lecture 4: More on soundness, compactness, completeness for PL

February 2, 2017 — Yinghao Wang

(These lecture notes are **not** proofread and proof-checked by the instructor.)

Reviews:

- Soundness: If a propositional formula has a tableau proof, then it is a tautology.
  $If \vdash A$, then $\models A$
  Soundness of a deductive system is the property that any sentence that is provable in that deductive system is also true on all interpretations or structures of the semantic theory for the language upon which that theory is based.
- Completeness: If a propositional formula is a tautology, then it has a tableau proof.
  $If \models A$, then $\vdash A$
  A formal system is called complete with respect to a particular property if every formula having the property can be derived using that system, i.e. is one of its theorems; otherwise the system is said to be incomplete.
- Compactness: If T is finitely satisfiable, then T is satisfiable.
  It provides a useful method for constructing models of any set of sentences that is finitely consistent.
- We should consider those logic problems as computer scientists. Therefore, as professor Kfoury states, we learn $Soundness + Completeness$ and then $Compactness$, instead of directly turn into $First - Order$.

Soundness:
Handout 06

Prove Soundness by Course-of-values induction/Strong induction. Consider from simplest WFFs as base case and then define the inductive step.
On $k > 1$, where k is number of lines in a formal proof. For the base case, we consider $k = n = 1$. From a given sequent $\varphi_1 \vdash \psi$, we want to show $\varphi_1 \models \psi$. Such a sequent implies $\varphi_1 = \psi$. Hence, $\psi \models \psi$, which is the same as $\varphi_1 \models \psi$

$$k\begin{cases} \rule{6cm}{0.4pt} \\ \ldots \\ \rule{6cm}{0.4pt} \\ \varphi \\ \rule{6cm}{0.4pt} \\ \psi \\ \rule{6cm}{0.4pt} \\ \frac{\varphi\ \psi}{\varphi\land\psi} \\ \rule{6cm}{0.4pt} \\ \ldots \\ \rule{6cm}{0.4pt} \end{cases}$$

After the base case, we can process our inductive step for lines that $k \geq 2$ $(1 \leq k \leq n)$.

Morgan's Law:
Handout 08

$\neg(A \land B) \equiv \neg A \lor \neg B$
$\neg(A \lor B) \equiv \neg A \land \neg B$
More in the next lecture..

Compactness sample exercise:
Exercise 8, compactness.pdf page 3:

Question: We can restrict the logical connectives of propositional logic to $\neg, \lor, \land$. Assume the set $X = \{x_1, x_2, x_3, \ldots\}$ of propositional variables is countably infinite. Suppose we extend this syntax with two new connectives, denoted W and V, each taking as a single argument a countably infinite set of previously defined WFFs. The resulting syntax is one version of what is called the infinitary propositional logic. If is a countably infinite set of the form $\Gamma = \{\varphi_1, \varphi_2, \varphi_3, \ldots\}$, then:

$$\bigvee \Gamma = \varphi_1 \lor \varphi_2 \lor \varphi_3 \lor \ \cdots,$$

and similarly for $\bigwedge \Gamma$.

Define the syntax of the infinitary PL, preferably in an extended BNF (Backus-Naur form)

Hint:

$$\text{Infinite set} \begin{cases} \text{All nature numbers, } \aleph_0 \rightarrow \text{Countable infinite} \\ \text{Real numbers, } \rightarrow \text{Uncountable infinite} \\ \text{Rational numbers} \\ \text{Set of well formed sequences of parentheses} \end{cases}$$

In standard WFFs, propositional variables $\rightarrow$ sets of countable infinite. Subset of countable infinite set can be finite/infinite.

$\varphi_1 = \bigvee\{x_1, x_3, x_5 \ldots\}$ $|\varphi| = \aleph_0$
$\varphi_1 = \bigvee\{x_2, x_4, x_6 \ldots\}$ $|\varphi| = \aleph_0$
$\bigwedge\{\varphi_1, \varphi_2\}$ size $= \aleph_0 + \aleph_0 = \aleph_0$

References:

1. http://www.cs.bu.edu/faculty/kfoury/UNI-Teaching/CS512/AK_Documents_Past_Semesters/compactness.pdf

2. http://www.cs.bu.edu/faculty/kfoury/UNI-Teaching/CS512-Spring17/Lecture/HD06.proof-of-soundness-for-PL.pdf