CS 512 Formal Methods, Spring 2017	Instructor: Assaf Kfoury

Lecture 2: Semantics of Classical Propositional Logic

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(These lecture notes are **not** proofread and proof-checked by the instructor.)

Compactness:

- Compactness ⇒ Topological sorting works for DAGs.
- Knigs Lemma: a finitely-branching infinite tree always has an infinite path.
  Every node's branch is finite. However, the tree is still infinite.
- Topological sorting
  finite DAG: diagonal circle can find a path.
- Soundness(theory) and completeness(truth) are like two-side of one story.

Infinitary PL:

- WFF can be infinite
  $|\varphi| = \mathbb{N}_0$ - countably infinite. $\mathbb{N}_0$ is the first infinite.
- $\mathbb{N}_0 + \mathbb{N}_0 = \mathbb{N}_0$
- $\mathbb{N}_0.\mathbb{N}_0 = \mathbb{N}_0 \rightarrow$ explained by dove tailing: will have all the pairs for natural numbers
- If we want to prove $\mathbb{N}_0 * \mathbb{N}_0 \leftrightarrow \mathbb{N}_0$, we can't use induction, since induction builds up induction hypothesis based on a finite n.

Compare Natural Deduction with Truth-Table:

- De Morgan's Law — tautologies.
  In CS, we focus on implementing the methods.
- Natural Deduction:
  1. For De Morgan's First Law, we can use Natural-deduction, but it's complicating and frustrating for us to write and use. And we use $\neg\neg e$.
  2. For De Morgan's Second Law, slightly simpler.
  3. For De Morgan's Third Law, don't have to use $\neg\neg e$.
  4. For De Morgan's Fourth Law, use $\neg\neg e$. (the way to prove it using Natural-deduction in the lecture slides is not the only way to do, but you'll still need $\neg\neg e$.
- Truth Table:
  Simpler than Natural-deduction and nicer. Since we only have two variables, we only need four rows.
  In Truth Table, the number of columns is linear to the size of formulas.

- Notice: $\neg\neg e \equiv LEM \equiv PBC$, which are forbidden by intuitionistically Propositional Logic. No intuitionistically valid formal proofs of De Morgan's First and Fourth Laws. $\rightarrow$ Truth Table beats Natural-deduction in this round.
- Complexity: Natural-deduction is better than Truth Table.
  # of rows in Truth Table: 2^n for n variables   Explode!!!
  # of columns in Truth Table; proportion of sub formulas   Just OK.

Terms of PL WFFs:

- Terms
  1. Valid WFFs / tautologies: have soundness and completeness, can get by Natural-deduction.
     In Truth Table: last columns are all true. $\leftarrow$ tautology.
  2. Satisfiable WFFs:
     In Truth Table: last columns have at least one row true.
  3. Unsatisfiable WFFs / contradictions.
     In Truth Table: last columns are all false.
  4. Falsifiable WFFs
     In Truth Table: last columns have at least one row false.
- Relationships:
  1. Valid WFFs is part of Satisfiable WFFs.
  2. Negation of satisfiable WFF is NOT unsatisfiable WFF, but valid WFF.
  3. Negation of falsifiable WFF is NOT tautological WFF.
- Representation:

φ	$\neg\varphi$
T	F
T	F
.	.
.	.
F	T
T	F
T	F
.	.
.	.
T	F
$\downarrow$	$\downarrow$
falsifiable	satisfiable

Semantic Tabeaux / Analytic Tableaux

- Want to show φ is analytic, prove $\neg\varphi$ is not satisfiable WFF.
- Unsigned Rules of Analytic tableaux:

$$\frac{\varphi \vee \psi}{\varphi \mid \psi} \quad \frac{\neg(\varphi \vee \psi)}{\neg\varphi \text{ or } \neg\psi} \quad \frac{\varphi \wedge \psi}{\varphi \text{ or } \psi} \quad \frac{\neg(\varphi \wedge \psi)}{\neg\varphi|\neg\psi} \quad \frac{\varphi \rightarrow \psi}{\neg\varphi|\psi} \quad \frac{\neg(\varphi \rightarrow \psi)}{\varphi \text{ or } \neg\varphi} \quad \frac{\neg\neg\varphi}{\varphi}$$

- Example: $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$

line 1:	$\neg\varphi$		Root of tableaux
line 2:	$\neg(p \wedge q)$	(from line 1)	Not conjunction
line 3:	$\neg(\neg p \vee \neg q)$	(from line 1)	Not disjunction
line 4:	$\neg\neg p$	(from line 3)	
line 5:	$\neg\neg q$	(from line 3)	
line 6:	p	(from line 4)	
line 7:	q	(from line 5)	

——————————————————— still linear

line 8: $\neg p$	$\neg q$	(from line 2)	
line 9: $\bot$	$\bot$		

Closed

- $\bot$: find a WFF and its negation along a path
- computer: check if a path is closed($p\&\neg p$ both exist along a path), if every path is closed, then φ is tautology.
- Tableau is maximum —— tried every sub rule.
- Roots for tableaux could be a finite set of WFFs.
- Tableaux check satisfiability: take the negation of whatever you want to prove satisfiability, try to close it.
  $\neg\phi \rightarrow \bot$: no way to satisfiable, which means ϕ is tautology.
  Strict: tree as thin as possible. For each (formula) WFF, the corresponding expansion rules has been applied at most once on each path.
- Theorem:
  Given a set of a tableau $\Gamma = \{\varphi_1, \varphi_2, ..., \varphi_n\}$ satisfiable $\Longleftrightarrow$ the tableau starting from Γ cannot be closed(unsatisfiable).
- $|\varphi| = n$ – a finite Propositional Logic
  among all size n WFFs.
  say for all WFFs of size 22, $f(22)$, does it has a proportion of valid WFFs?
  If have fixed number of variables, at some point, more WFFs doesn't give more information (just repeating or negating of previous WFFs).

Next time:

- expandable to first order
- more tableaux
- Resolution
- DPN

Reference:

1. http://www.cs.bu.edu/faculty/kfoury/UNI-Teaching/CS512-Spring17/Lecture/HD07.universe-of-PL-WFFs.pdf
2. http://www.cs.bu.edu/faculty/kfoury/UNI-Teaching/CS512-Spring17/Lecture/HD08.de-morgans-laws.pdf