Scribe Notes

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(These lecture notes are not proofread or proof-checked by the instructor.)

1 First Order Logic

First order logic adds quantifiers ($\forall$ and $\exists$) and variables to propositional logic.

For example: for all x, if x is a bird then x has wings

Can be modeled using the variable x, a function $B(x)$ and $W(x)$ where $B(x)$ means that x is a bird and $W(x)$ means that x has wings. We can now write this more succinctly as:

$$\forall x(B(x) \to W(x))$$

Let's now discuss the three parts of first order logic, vocabulary, terms and WFF's. (Note that on wikipedia vocabulary is known as the alphabet, terms are terms and WFF's are just formula's, you can read more about them here `https://en.wikipedia.org/wiki/First-order_logic`)

1.1 Vocabulary

- A set $\mathcal{P}$ of predicate symbols with arity $n \geq 0$
  - note: arity is the number of parameters that function accepts so a function $f(x)$ has arity 1 and $f(x, y)$ 2.

- note: Usually denoted with upper case variables such as P or with arity 1 $P(x)$.
- note: if this confuses you read about it more here: `https://en.wikipedia.org/wiki/First-order_logic#Non-logical_symbols`

For example: $P(x)$ is a predicate variable of arity 1. One possible interpretation is "x is a man".

- A set $\mathcal{F}$ of function symbols with arity $n \geq 1$

  For example: $f(x)$ may be interpreted as for "the father of x". In arithmetic, it may stand for $-x$. In set theory, it may stand for "the power set of x". In arithmetic, $g(x, y)$ may stand for $x + y$. In set theory, it may stand for "the union of x and y". (from wikipedia)

- A set $\mathcal{C}$ of constant symbols with arity $n = 0$

  For example: c is a constant symbol that means false. a is a constant symbol true.

1.2 Terms

- a variable x is a term
- a constant $c \in \mathcal{C}$ is a term
- if $t_1, \ldots, t_n$ are terms and $f \in \mathcal{F}$ is a n-ary function then $f(t_1, \ldots, t_n)$ is a term

BNF

$$t ::= x|c|f(t, \ldots, t)$$

Note: Note that the above definitions refer to the $\mathcal{F}$,$\mathcal{C}$ and $\mathcal{P}$ defined in section 1. Which in term refer to definitions that we previously defined for function symbols, predicate symbols and constant symbols. If all of this is very confusing (as it was to me) see this section on wikipedia page `https://en.wikipedia.org/wiki/First-order_logic#Formation_rules`

1.3 Well Formed Formulas

- if $t_1, \ldots t_n$ are terms and $P \in \mathcal{P}$ then $P(t_1, \ldots, t_n)$ is a WFF
- if t_1, t_2 are terms then $t_1 \doteq t_2$ is a WFF (note: $\doteq$ is equality within formal language)
- if φ is a WFF then so is $\neg\varphi$
- if φ and ψ are WFF's then so are $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$ and $(\varphi \rightarrow \psi)$
- if φ is a WFF and x is a variable, then $(\forall xp)$ and $(\exists x\varphi)$ are WFF's

BNF

$\varphi ::= P(t_1, \ldots, t_n)|t_1 \doteq t_2|(\neg\varphi)|(\varphi \vee \varphi)|(\varphi \wedge \varphi)|(\varphi \rightarrow \varphi)|(\forall x\varphi)|(\exists x\varphi)$

1.4 Are all WFF's of propositional logic WFF's of predicate logic (1st order logic) ?

You may be thinking that propositional logic is a subset of first order logic, this isn't strictly true as a valid WFF in first order logic cannot have free standing propositional variables. To see why consider the following BNF definitions.

$$t ::= x|c|f(t, \ldots, t)$$

$$\varphi ::= P(t_1, \ldots, t_n)|t_1 \doteq t_2|(\neg\varphi)|(\varphi \vee \varphi)|(\varphi \wedge \varphi)|(\varphi \rightarrow \varphi)|(\forall x\varphi)|(\exists x\varphi)$$

You'll notice that the constant c (which is a propositional variable) that we're after cannot be a valid WFF φ because there's nothing in the BNF like $\varphi ::= t$.

So to "imbed" propositional logic in 1st order logic we have to use a function f that I'm going to call the identity function.

$$f(x) = x$$

You'll notice that this is simply a wrapper to satisfy the BNF definition so that we can have raw terms in WFF's φ

1.5 Transitive

$$\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$$

1.6 Reflexive

$$\forall x R(x, x)$$

1.7 Symmetric

$$\forall x \forall y (R(x, y) \rightarrow R(y, x))$$

1.8 Notation

We're learning some new notation which confused me so I thought I'd jot down the notation here:

$$\varphi[x/t]$$

This means replace x with t in φ, it's also referred to as $\varphi[x \rightarrow t]$ and $\varphi[x := t]$.

$$R(x, x)$$

This is "prefix notation as opposed to "infix notation which looks like:

$$x \doteq x$$