CS 512 Formal Methods, Spring 2014	Instructor: Assaf Kfoury
First-Order Formulas	
March 21, 2017	Justin Chen

(These lecture notes are **not** proofread and proof-checked by the instructor.)

1 Administrative

- Assignment 7 is posted and due March 31st.
- 3 new handouts have been posted

2 First-Order Relational Structures

2.1 Order

- Has to be over same signature over same vocabulary
- First-order structure aka first-order relational structure
- Signature is usually given with order
- Signature consists of predicate symbols, function symbols

e.g. $\mathcal{M}_1 = (\mathbb{N}, \doteq \mu_1, \leq \mu_1, ...)$
contains a smallest element.
e.g. $\mathcal{M}_2 = (\mathbb{Z}, \doteq \mu_2, \leq \mu_2, ...)$
does not contain a smallest element.
e.g. $\mathcal{M}_3 = (\mathbb{R}, \doteq, \leq, ...)$
e.g. $\mathcal{M}_4 = (\mathbb{C}, \doteq, ...)$
ordering on complex structures
e.g. $\mathcal{M}_5 = (\mathcal{P}(s), \doteq, \subseteq, \cup, \cap, \emptyset)$
S is an arbitrary set. P is the power set.
e.g. $\mathcal{M}_6 = (r \in \mathbb{R} | 0 \leq r \leq 1, \doteq, \leq, +, -, \cdot, /)$
Dense linear has a smallest and largest element. *Brief digression*:

Singleton: set with one thing
Doubleton: set with two things

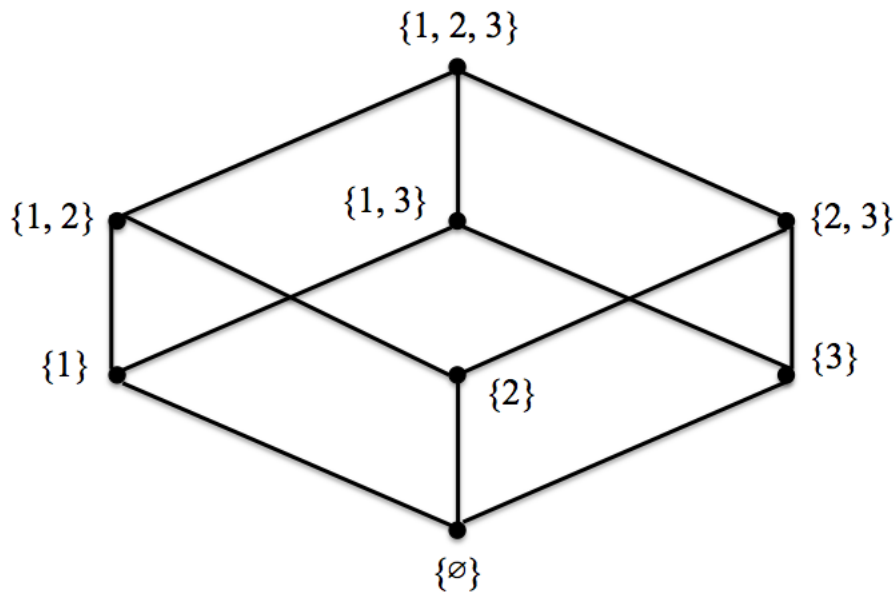

Figure 1. Graphical representation of relationship between empty set, singletons, doubletons, and tripletons. Refer to slide titled *Orders* in *Handout 18.*

2.2 Graphs

- Algebras with two binary operations:
- Can interpret the ”xor” symbols in the ”structure” of M6

$\mathcal{G} = (V, \doteq, \mathcal{R}^G)$
e.g. binary relation $\mathcal{R}^G(a, b)$, $\mathcal{R}^G(a, c)$, $\neg\mathcal{R}^G(a, d)$, etc.

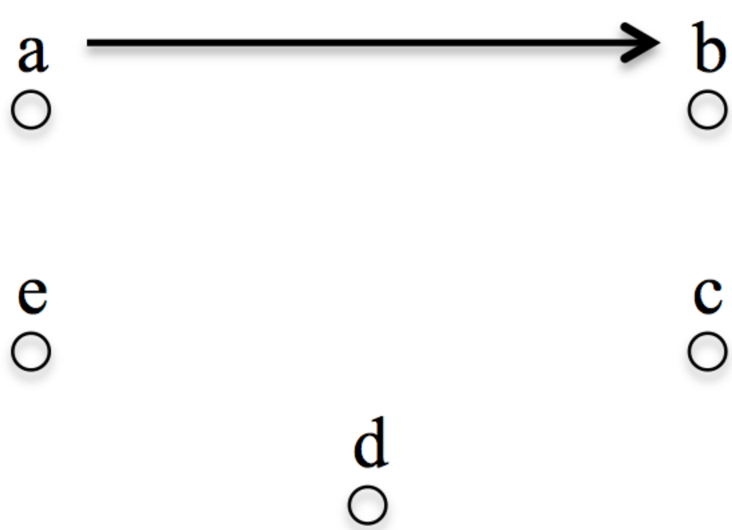

Figure 2. Refer to slide titled *Graphs* in *Handout 18.*

e.g. What does this say?
$\forall x \forall y(\neg(x \doteq y) \rightarrow (\mathcal{R}(x, y) \rightarrow \mathcal{R}(x, y)))$

Don't really need the premise above if you have formula 2 $x(\neg\mathcal{R}(x,x))$ since it's redundant.
There are no loops.
Structures in multiple domains(universes):
Capacitated Graph: $\mathcal{G}(V, \mathbb{R}; \dot{=}, \mathbb{R}, c)$

- Domains are for V, and R sets of functions
- c - capacity function (upperbound)
- Disjoint sets of functions can't be subsets
- ; indicates separate universes
- Can't have multiple capacities
- Similar to Max-Flow
- $C : V \times V \to \mathbb{R}$

e.g.

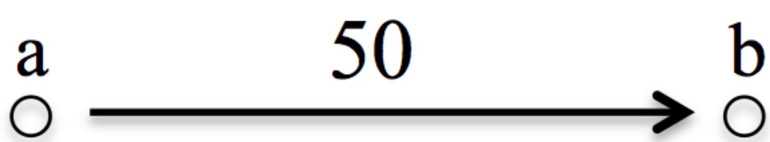

Figure 3.

Successor function over the natural numbers

- 4 is simple induction
- phi(x) is your first-order formula/structure, your "property"
- phi(0) is the basis step
- thing to the right before arrow is induction step
- implicitely a larger parens around the expression to the left of arrow
- phi(x) is induction hypothesis
- phi(x) -¿ phi(Sx) is induction step
- right of outside most arrow says this proves for all phi(y)
- property must be first-order expressible, but not all properties are first-order expressible
- can add these formulas as axioms
- How do you know if you use this in your knowledge base, that it's a correct way of reasoning?
- Must show consistency, will explain more later

3 is not first-order because quantifying over ground-value, which is not in the universe.
Second-order formula: $\forall x \forall y (f(x,y) \leq c(x,y))$
e.g.Not a first-order relational structure
$(V, \mathbb{R}, (R)^{V \times V}, ...)$

- Universals must precede existentials
- If e.g. $\exists y \forall x_1 \forall x_2 (...)$ then $\rightarrow \exists x \exists y \exists z ... x ... y ... z .. xz ...$
- so drop $\exists x$ and replace x with constant c
- Herbrand Theory
- - Gives a unified framework
- - Each of above gives tools in first-order that you can implement
- – Tableaux
- – Resolution
- – SMT
- Compactness slide page 11
- - phi and phi1 are the same
- - prenex or prenex normal form
- - notice that quantifiers are in different positions
- - ϕ_1 factors the quantifiers out and places it to the left
- - in CNF, move logical and or logical or out and push logical negation in, and flip
- - DeMorgan's law
- - same happens with quantifiers universal and existential
- Prenex of conjuctive normal form: ϕ_2

2.3 Skolem

e.g. $\forall x \exists y (...x, y, x, ..., y, ...)$

- $\forall x$ - adversary tries to falsify by choosing x
- $\exists y$ - agent tries to make satisfiable by choosing y

Agents move depends on what adversary does, by saying "depends" implies Skolem function
Universal WFF:
$\forall x (...x...f(x)....x...f(x)...)$
e.g.
$\forall x_1 \forall x_2 \exists y (...x_1...x_2...y...x_1...y...x_2...y)$
$\forall x_1 \forall x_2 (...x_1...x_2...f(x_1, x_2)...x_1...f(X_1, x_2)...x_2...f(x_1, x_2...)$