CS 512 Formal Methods, Spring 2017	Instructor: Assaf Kfoury

Soundness and completeness of first-order logic

March 28, 2017 — Yinghao Wang

(These lecture notes are **not** proofread and proof-checked by the instructor.)

- **Miscellanea**:
  Homework 7 due on Friday, Mar.31^{st}.
  Homework 8 out on Thursday, Mar.30^{th}.

- **Readings**: Handout 21, Handout 22

- $\mathcal{Z} = (\mathbb{Z}, =, <, +, \times, 0, 1)$

- $\varphi \triangleq \forall x \exists y (x + y \doteq 0)$
  -Prenex normed form
  -not skolemized.

- If φ is true/satisfiable by $\mathcal{Z}$, we write $\mathcal{Z} \models \varphi$

- It is possible to refer to elements in the universe without "naming them".

  1. $0^{\mathcal{Z}}$ is zero
  2. $1^{\mathcal{Z}}$ is one
  3. $(1+1)^{\mathcal{Z}}$ is two

  We have expression to write all non-negative integers.

- for $\varphi \triangleq \forall x \exists y (x + y \doteq 0)$

  1. Skolemized φ: $\varphi' \triangleq \forall x (x + f(x) \doteq 0)$
  2. Ground terms:
     - $0, 1, 1+1, 1*1, (1*1)+1$
     - $f(0), f(1+1), f(1*1), f((1*1)+1)$
  3. Signature has been argumented with $f(\ )$
  4. In here, $f()$ maps x to $-x$
  5. $f(1 + f(1)) = f(1 + (-1)) = f(0) \to 0$

- $\mathcal{R} \triangleq (\mathbb{R}, =, <, +, \times, 0, 1)$ field*

- for $\exists! x \varphi(x)$: there exist a unique x such that $\varphi(x)$
  $\exists x(\varphi(x) \wedge \forall y(\neg(x \doteq y) \rightarrow \neg\varphi(y)))$
  - $\exists^{=3} x \varphi(x)$: there exists exactly 3
  - $\exists^{\geqslant 3} x \varphi(x)$: there exists at least 3

  1. $\mathcal{R} \not\models \forall x \exists! y(y \times y \doteq x)$ Example: $2 \times 2 = 4$ and $-2 \times -2 = 4$
  2. $\mathcal{R} \not\models \forall x \exists^{=2} y(y \times y \doteq x))$ Example: if $x < 0$, there is no such y.
  3. $\mathcal{R} \models \forall x \exists^{=2} y(0 < x \rightarrow y \times y \doteq x)$
  4. $\mathcal{R} \models \forall x \exists! y(0 < x \rightarrow (0 < y \wedge (y \times y \doteq x))$
  5. Skolemized: $\varphi' \triangleq \forall x(0 < x) \rightarrow (0 < f(x) \wedge (f(x) \times f(x) \doteq x))$
     (a) φ and φ' are equally satisfiable
     (b) $\{f\} \bigcup \{=, <, +, \times, 0, 1\}$
     (c) Considering the ground term, this is only for natural numbers and square roots of nature numbers.

- Consistency:
  - Γ is a set of WFFs,
  - Γ is consistent iff $\Gamma \nvdash \perp$
  - if Γ is satisfiable, then Γ is consistent.