CS 512 Formal Methods, Spring 2014	Instructor: Assaf Kfoury
Herbrand Theory and Gilmore's Algorithm	
April 4, 2017	Scribe: Johnson Lam

(These lecture notes are **not** proofread and proof-checked by the instructor.)

1 Herbrand Theory and Review

Please reference the Compactness and Completeness handout for a detailed proof.

- Lemma 21: If ϕ is a first-order sentence, then ϕ and $\theta_{pr,sk}(\phi)$ are equisatisfiable
- Lemma 28: If ϕ is a first-order sentence, then ϕ and the Herbrand expansion $GR_E expansion(\theta_{pr,sl}(\phi))$ are equisatisfiable.
- Theorem 32: If Γ is a set of first-order sentences, finite of infinite, and $\Gamma^{'} = Gr\_Expansion(\theta_{pr,sk}(\Gamma))$ is the corresponding set of quantifier-free sentences, then the following assertions are true:
  1. Γ and $\Gamma^{'}$ are equisatisfiable (first-order logic).
  2. If Γ is finitely satisfiable, then $\Gamma^{'}$ is finitely satisfiable (first-order logic).
  3. $\Gamma^{'}$ (first-order logic) and $\chi(\Gamma^{'})$ (propositional logic) are equisatisfiable.
  4. $\Gamma^{'}$ (first-order logic) and $\chi(\Gamma^{'})$ (propositional logic) are finitely equisatisfiable.

1.1 In-class Example

Let p be a unary predicate symbol and f be a unary function symbol and let f be written as the following:

$$f(x) := fx$$
$$f(f(fx)) := f^3x$$

The following is a universal first-order sentence:

$$\theta_{pr,sk}(\phi) \triangleq \forall x(p(x) \land \neg p(f(x)))$$

Prenex: $\forall x$
Matrix: $p(x) \land \neg p(f(x))$

To derive the Herbrand Expansion, We now generate the set of ground terms of ϕ, delete the prenex, and substitute the ground terms for variables in the matrix to obtain the ground expansion- we include a constant c because there is no constant in ϕ.

Let the ground terms be

$$Gr\_Terms(\phi) = \{c, fc, f^2c, ...\}$$

Then the ground expansion will be the following

$$Gr\_Expansion(\theta_{pr,sk}(\phi)) =$$
$$Gr\_Expansion(\phi) =$$
$$= \{p(c), \neg p(fc), p(fc), \neg p(f^2c), ...\} \cup \{t_1 = t_2 | t_1, t_2 \in Gr\_Terms(\phi)\}$$

To generate the propositional logic form, consider the following:

$$\begin{aligned}\chi(Gr\_Expansion(\theta_{pr,sk}(\phi))) &= \\ \chi(Gr\_Expansion(\phi)) &= \\ &= \{\chi_{p(c)}, \chi_{\neg p(fc)}, \chi_{p(fc)}, \chi_{p(f^2c)}, ...\}\end{aligned}$$

2 Gilmore's Algorithm

Goal: Test whether a first-order setence is valid
Input: first order setence ϕ
Output: true or false if ϕ is valid

1. $k = 0$
2. repeat: $k = k + 1$
   $\bigwedge_{1 \leq i \leq k} \theta i$
3. return results

Note

- Gilmore's Algorithm terminates iff the input sentence ϕ is valid, making it a semi-decision procedure
- Gilmore's Algorithm is highly inefficient and it's performance depends ont he order in which the θ_i's are generated

2.1 In-class Example

Let t be a ground term:

$$\frac{\forall x \phi}{\phi[x = t]} \text{ or } \frac{\neg \exists x \phi}{\neg \phi[x = t]}$$

Let c be a fresh constant symbol:

$$\frac{\exists x \phi}{\phi[x = c]} \text{ or } \frac{\neg \forall x \phi}{\neg \phi[x = c]}$$

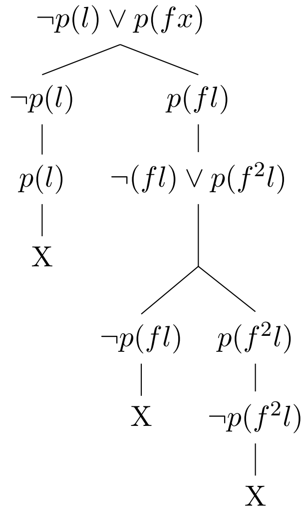
The following is a set of first order sentences:

$$\Gamma = \{\forall x(\neg p(x) \wedge p(fx)), p(l), \neg p(f^2 l)\}$$

The ground terms are:

$$Gr\_Terms(\Gamma) = \{l, fl, f^2 l, ...\}$$

The following tableaux tree, substituted with the ground terms, shows that the set Γ is unsatisfiable:


$\neg p(l) \vee p(fx)$
$\neg p(l)$
$p(fl)$
$p(l)$
$\neg(fl) \vee p(f^2l)$
X
$\neg p(fl)$
$p(f^2l)$
X
$\neg p(f^2l)$
X