CS 512 Formal Methods, Spring 2017 Instructor: Assaf Kfoury

Limits of Formal Modeling in Propositional Logic and First-Order Logic

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(These lecture notes are **not** proofread and proof-checked by the instructor.)

1 Two Kinds of Limits

- **Complexity limits**
- **Expressiveness limits**

2 The Pigeon Hole Principle

For more detail about The Pigeon Hole Principle, back to handout "Formal Modeling with Propositional Logic".

The Pigeon Hole Principle

For every natural number $n > 2$, the Pigeon Hole Principle (PHP) states: If n pigeons sit in $(n-1)$ holes, then some hole contains more than one pigeon. We want to formalize PHP in propositional logic (PL). There are different ways of doing this, but perhaps the most natural is:

Use propositional atom $P_{i,j}$ to indicate that pigeon i is in hole j, where $1 \leqslant i \leqslant n$ and $1 \leqslant j < n$. With this formal representation of *pigeon i is in hole j* we can formalize PHP with the following PL formula φ:

$$\varphi \triangleq \bigvee_{1 \leqslant i \leqslant n} (\bigwedge_{1 \leqslant j < n} P_{I,J}) \rightarrow \bigvee_{1 \leqslant < k \leqslant n} (\bigvee_{1 \leqslant j < n} (P_{i,j} \wedge P_{k,j})) \tag{1}$$

where $\bigvee$ and $\bigwedge$ are shorthand notation to write long sequences of conjunctions and disjunctions, respectively.

We now want a first-order sentence Ψ in the signature $\sum = R, c$where R is a binary relation symbol and c is a constant symbol, such that:

Every structure Mn of the form $(\{1, 2, ..., n\}, R^{\mathcal{M}_n}, c^{\mathcal{M}_n})$ is a model of Ψ and the interpretation of Ψ in Mn expresses PHP$_n$.

Here is a possible first-order formulation of Ψ:

$$\Psi \triangleq (\forall x \exists y R(x,y)) \wedge (\forall x \neg R(x,c)) \rightarrow \exists v \exists w \exists y (\neg(v \doteq w) \wedge R(v,y) \wedge R(w,y)) \tag{2}$$

where v, w are distincitive variables.

e.g we can use graph to present the above formulation

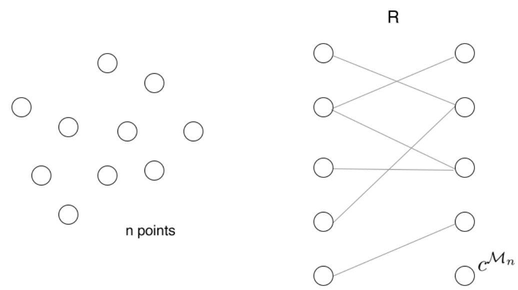

Figure 1:

Exercise: Translate Ψ into a propositional WFF n which depends on an additional parameter $n > 2$. (Ψ represents an infinite family of propositional WFFs, one ψ_n for every $n > 2$.)

Hint: Consider replacing every $\forall$ by a $\bigwedge$ and every $\exists$ by a $\bigvee$.

$$\psi_n \triangleq (\bigwedge_{1 \leqslant i \leqslant n} \bigvee_{1 \leqslant j < n} \mathcal{P}_{ij}) \wedge (\bigvee_{1 \leqslant i \leqslant n} \neg \mathcal{P}_{in}) \rightarrow (\bigvee_{1 \leqslant i \leqslant n} \bigvee_{1 \leqslant j \leqslant n} \bigvee_{1 \leqslant k \leqslant n} (\mathcal{P}_{ik} \wedge \mathcal{P}_{jk})) \tag{3}$$

For The Pigeon Hole Principle, in particular, for the case n = 3, we get:

$$\begin{aligned} \varphi_3 \triangleq (\mathcal{P}_{11} \wedge \mathcal{P}_{12}) \vee (\mathcal{P}_{21} \wedge \mathcal{P}_{22}) \vee (\mathcal{P}_{31} \wedge \mathcal{P}_{32}) \rightarrow \\ (\mathcal{P}_{11} \wedge \mathcal{P}_{21}) \vee (\mathcal{P}_{12} \wedge \mathcal{P}_{22}) \vee (\mathcal{P}_{11} \wedge \mathcal{P}_{31}) \vee (\mathcal{P}_{12} \wedge \mathcal{P}_{32}) \vee (\mathcal{P}_{21} \wedge \mathcal{P}_{31}) \vee (\mathcal{P}_{22} \wedge \mathcal{P}_{32}) \end{aligned}$$

By comparing φ_n and Ψ_n, we can get the following facts:

- A resolution proof of φ_n or Ψ_n is possible but does not help
  More precisely, any resolution proof of φ_n or Ψ_n has size at least $\Omega(2^n)$
- There are proofs of φ_n and Ψ_n using what is called extended resolution (not covered this semester) which have size $\mathcal{O}(n^4)$.
- : There are Hilbert-style proofs (not covered this semester) of φ_n and Ψ_n which have size at most $\mathcal{O}(n^{20})$.

3 How strong is first-order logic?

Two similar first-order sentences:

$$\theta_1 \triangleq \forall x \exists y (x < y \wedge prime(y) \wedge prime(y+2)) \tag{4}$$

$$\theta_2 \triangleq \forall x \exists y (\neg(x \doteq 0) \rightarrow (x < y) \wedge (y \leqslant 2 \times x) \wedge prime(y)) \tag{5}$$

both to be interpreted in the structure $\mathcal{N} \leqslant (\mathcal{N}; \times, +, 0, 1)$ and where $prime()$ is a unary predicate that tests whether its argument is a prime number.$prime()$ is first-order definable in $\mathcal{N}$.

θ_1 formally expresses the *Twin-Prime Conjecture*, a long-standing open problem.

θ_2 formally expresses the Bertrand-Chebyshev Conjecture, which was shown to be true by hand, before digital computers were invented

Bertrand-Chebyshev Conjecture (Cite from Wikipedia)

Bertrand's postulate is a theorem stating that for any integer $n > 3$, there always exists at least one prime number p with

$$n < p < 2n - 2 \tag{6}$$

Another formulation, where p_n is the n-th prime, is for $n \geq 1$

$$p_{n+1} < 2p_n \tag{7}$$

This statement was first conjectured in 1845 by Joseph Bertrand (18221900). Bertrand himself verified his statement for all numbers in the interval $[2, 3 \times 10^6]$. His conjecture was completely proved by Chebyshev (18211894) in 1852 and so the postulate is also called the BertrandChebyshev theorem or Chebyshev's theorem. Chebyshev's theorem can also be stated as a relationship with $\pi(x)$,, where $\pi(x)$is the prime counting function (number of primes less than or equal to x:

$$\pi(x) - \pi(\tfrac{x}{2}) \geq 1, \; for \; all \; x \geq 2, \tag{8}$$

Theorem: Skolem-Lowenheim

1. If φ is a first-order sentence such that, for every ≥ 1, there is a model of φ with at least n elements, then φ has an infinite model.

First-order logic cannot enforce finiteness of models.

2. If φ is a first-order sentence which has a model (i.e., φ is satisfiable), then φ has a model with a countable universe.

First-order logic cannot enforce uncountable models.

Theorem

There is no first-order WFF $\psi(x, y)$ with two free variables x and y, over the signature $\{R, \doteq\}$ where R is a binary predicate symbol, such that for every graph model $\mathcal{M} = (M, R_{\mathcal{M}})$ and every $a, b \in \mathcal{M}$, it holds that:

$$\mathcal{M}, a, b| = \psi \; iff \; there \; is \; a \; path \; from \; a \; to \; b \tag{9}$$

Reachibility in graphs is not first-order definable.