Scribe Notes

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(These lecture notes are **not** proofread or proof-checked by the instructor.)

1 Motivation

In these notes I'm examining several problems that can be modeled with second oder logic. Second order logic adds quantification over predicate symbols and function symbols. An example of this is as follows:

$$\exists P \forall x (P(x))$$

Where P is a unary predicate symbol and x is a variable.

2 Isomorphism

Let's first consider an example of isomorphism. Take the structures $\mathcal{N}^D$ and $\mathcal{N}^B$ where B stands for binary and D is decimal.

$$\mathcal{N}^D \triangleq (\mathbb{N}, \times, +, 0, 1)$$

$$\mathcal{N}^B \triangleq (\{0, 1, 10, 11, 100, \dots\}, \times^B, +^B, 0^B, 1^B)$$

You'll notice that they are exactly the same, the binary version is just another way of representing the decimal version. These structures are not equivalent $\mathcal{N}^D \neq \mathcal{N}^B$ but we say that they're isomorphic, the mathematical notation for this is $\mathcal{N}^D \cong \mathcal{N}^B$.

3 Formal Definition of Isomorphism

Let $\mathcal{M}$ and $\mathcal{N}$ be structures with the same signature σ and universe $\mathcal{M}$ and $\mathcal{N}$, respectively $\mathcal{M}$ and $\mathcal{N}$ are **isomorphic** $\mathcal{M} \cong \mathcal{N} / \mathcal{M} \equiv \mathcal{N}$ iff there is a bijection (one-to-one)

$$\theta : \mathcal{M} \to \mathcal{N}$$

such that

1. for every predicate symbol $P \in \Sigma$ of arity $n \geq 0$ and all $a_1, a_2, \ldots, a_n \in M$

$$(a_1, a_2, \ldots, a_n) \in P^M \text{iff} \Big(\theta(a_1, \ldots, \theta(a_n)\Big) \in \mathcal{P}^{\mathcal{N}}$$

2. for every function symbol $f \in \Sigma$ of arity $n \geq 1$ and all $a_1, \ldots a_n \in \mathcal{M}$

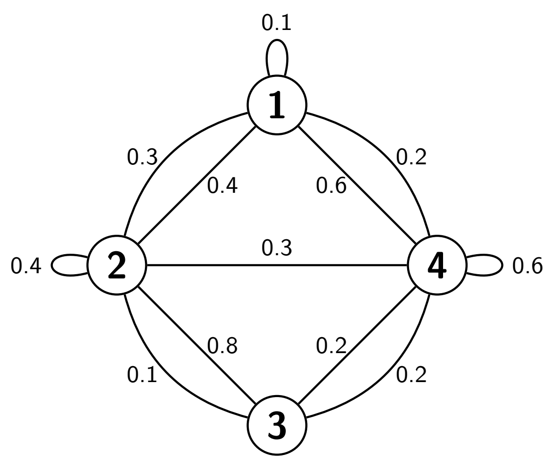
$$\theta(f^{\mathcal{M}}(a_1, \ldots a_n)) = f^{\mathcal{N}}(\theta(a_1, \ldots, \theta(a_n))$$

3. for every constant symbol $c \in \Sigma$,

$$\theta(c^{\mathcal{M}}) = c^{\mathcal{N}}$$

4 Hamiltonian Path

Next we're going to look at a Hamiltonian Path, you'll remember that a Hamiltonian path is a path that visits every node in a graph exactly once. In the below graph, a Hamiltonian path could go from node 1 to 4 to 3 to 2.



Let's say we have set of nodes A and a binary relation R such that $R(x, y)$ means there's an edge from $x \in A$ to $y \in A$.

This is expressible in second order like so:

$$\varphi \triangleq \exists P\Big(\text{"P is a linear order"} \wedge \forall x \forall y (y = x + 1 \rightarrow R(x, y))\Big)$$

We can simplify this by adding a WFF ψ_1 that determines if P is a linear order.

$$\varphi \triangleq \exists P\Big(\psi_1(P) \wedge \forall x \forall y (y = x + 1 \rightarrow R(x, y))\Big)$$

In one of the handouts we defined ψ_1, it is reproduced below for convenience. Remember that $\psi_1(P)$ means that predicate-variable P has linear order.

$$\begin{aligned}
\psi_1(P) \triangleq & \forall x P(x, x) && \text{(reflexivity)}\\
& \wedge \forall x \forall y \forall z \Big(P(x, y) \wedge P(y, z)) \rightarrow P(x, z)\Big) && \text{(transitivity)}\\
& \wedge \forall x \forall y \Big(P(x, y) \wedge P(y, x) \rightarrow x \doteq y\Big) && \text{(anti-symmetry)}\\
& \wedge \forall x \forall y \Big(P(x, y) \wedge P(y, x)\Big) && \text{(totality)}
\end{aligned}$$

5 Example: Two Coloring

Let's consider the example of coloring a graph using two colors such that any node does not share the same color as any node that it has an edge with. Alternatively you can think of a map of the world. The coloring problem is how do you color the countries in the world map such that no two adjacent countries have the same color. Two coloring a specific version of the n-coloring problem where you only have two colors.

We know that when $n \geq 3$ the problem is NP-Complete.

Let's represent one color by unary predicate P and the other by $\neg P$. Now we can model the problem using a second order WFF:

$$\varphi \triangleq \exists P \forall x \forall y \Big(\neg(x \doteq y) \wedge R(x, y) \rightarrow (P(x) \leftrightarrow \neg P(y))\Big)$$

6 Example: Three Coloring

Remember that three coloring is simply coloring each node in a graph such that no adjacent nodes have the same color using only three colors.

Let's create a WFF ψ_1 that means "Each node has exactly one color".

$$\begin{aligned}
\psi_1(A_1, A_2, A_3) \triangleq \forall x \Big(& (A_1(x) \wedge \neg A_2(x) \wedge \neg A_3(x))\\
& \vee (\neg A_1(x) \wedge A_2(x) \wedge \neg A_3(x))\\
& \vee (\neg A_1(x) \wedge \neg A_2(x) \wedge A_3(x))\Big)
\end{aligned} \tag{1}$$

You'll notice we can no longer use P and $\neg P$ because we have more than two colors. Instead we need to introduce three unary predicates $A_1, A_2, \ldots, A_3$ that indicate the color of each node x.

We need to introduce another WFF ψ_2 that means "no two points with the same color are connected".

$$\begin{aligned} \psi_2(A_1, A_2, A_3) \triangleq \forall x \forall y \Big(&(A_1(x) \land A_1(x) \rightarrow \neg R(x, y)) \\ &\land (A_2(x) \land A_2(x) \rightarrow \neg R(x, y)) \\ &\land (A_3(x) \land A_3(x) \rightarrow \neg R(x, y)) \end{aligned} \tag{2}$$

Now for our final WFF φ we can combine the two previous WFF's and form the second order sentence.

$$\varphi \triangleq \exists A_1 \exists A_2 \exists A_3 (\psi_1 \land \psi_2)$$

7 Example: Un-connectedness

Now we turn our attention to un-connectedness. A graph is unconnected if there exists a set of nodes that do not have edges connecting them to the other nodes. Let's say we have a set of nodes A, we can write a WFF φ by combining two WFF's ψ_1 and ψ_2 like so:

(a) "The set A is non-empty and its complement is nonempty". Remember complement in graph theory just means the set of nodes that are not in A. You can think of the predicate P as returning true if a node $P(x)$ exists and false otherwise.

$$\psi_1 = \exists x \exists y \Big(P(x) \land \neg P(y) \Big)$$

(b) "There is no edge between A and its complement"

$$\psi_1 = \exists x \exists y \Big((P(x) \land \neg P(y)) \rightarrow (\neg R(x, y) \land \neg R(y, x)) \Big)$$

$$\varphi \triangleq \exists A (\psi_1 \land \psi_2)$$

This is only true if A is unconnected.

8 Example

Suppose we have a signature $\Sigma = (\dots, f, \dots)$ where f is a unary function symbol.

Consider the first order sentence:

$$\varphi \triangleq \forall x\Big(\neg(fx \doteq x) \land \forall x \forall y(fx \doteq y \leftrightarrow f(y) \doteq x)\Big)$$

$\mathcal{M} = (\mathcal{M}, f^{\mathcal{M}})$

Suppose that $\mathcal{M}$ is infinite. What can we say about the size?

Is it even or odd? What can we say about the polarity of $\mathcal{N}$?

- If the $|\mathcal{M}| \mod 2 = 0$ (the size of $\mathcal{M}$ is even) $\mathcal{M} \not\models \varphi$
- if $|\mathcal{M}$— is even then $\mathcal{M} \models \varphi$
- if $\mathcal{M} \models \Phi$ and $|\mathcal{M}|$ is finite, then $|\mathcal{M}|$ is even.

We can rewrite this in second order like so:

$$\varphi \triangleq \exists F\Big(\neg(Fx \doteq x) \land \forall x \forall y(Fx \doteq y \leftrightarrow Fy \doteq x)\Big)$$