Lecture Notes
Compactness and Completeness of Propositional Logic and First-Order Logic

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In these notes I follow a recent trend of introducing and proving the Compactness Theorem before the Completeness Theorem. Doing it this way, Completeness becomes a consequence of Compactness. The other way around, which is standard in many textbooks on mathematical logic and formal methods, invokes Completeness (as well as Soundness) to prove Compactness .[1]

There are good reasons for reversing the traditional approach. Perhaps the chief reason is to avoid getting immersed in nitty-gritty details of a formal proof system and, thus, to also avoid the concern of dealing with as many proofs of Completeness as there are proof systems (a welcome avoidance when our study is to compare different proof systems for propositional logic and first-order logic).

Another reason, no less compelling, for starting with Compactness is didactic: It makes it easier to grasp topological aspects of the notion (and the origin of its name). We can go deep in a topological direction, by explaining Compactness purely in terms of notions such as *ultrafilters*, *ultraproducts*, *Hausdorff spaces*, the *finite-intersection property*, and others, but that would take us further afield from the focus of this course.[2]

I choose a watered-down topological approach. In the proof of Compactness in Section 1, we construct a maximal satisfiable set of propositional WFF's and avoid explicitly defined topological notions, although some of these are lurking right under the surface.[3]

Another uncommon aspect in these notes is to reduce Compactness for first-order logic to Compactness for propositional logic.[4] Alternative and more common approaches to proving Compactness for first-order logic – *whether* first and directly, *or* second and as a consequence of Completeness and Soundness – do not need to invoke these properties in relation to propositional logic. By contrast, the proof of Compactness for first-order logic in these notes (Section 5) requires an explicit invocation of Compactness for propositional logic via what is called *Herbrand theory* (in Section 4).

[1]A typical example is the proof of the Compactness Theorem in Enderton's book, *A Mathematical Introduction to Logic*, at the end of Section 2.5. That proof invokes the Completeness Theorem, as well as the Soundness Theorem, to prove Compactness. The book I recommended as a reference for this course (M. Huth and M. Ryan, *Logic in Computer Science*, Second Edition, Cambridge University Press, 2004) omits altogether the proofs for Soundness and Completeness (on page 96), and then invokes Completeness to prove Compactness as a consequence (on page 137).

[2]If you are interested in knowing more, search the Web for "topological proof of the compactness theorem in logic".

[3]One such notion is the *finite-intersection property*: A collection $\mathcal{A}$ of subsets of a set X is said to have the finite-intersection property if the intersection over any finite subcollection of $\mathcal{A}$ is nonempty.

[4]I do not claim originality for this approach. It originated with others. For example, this approach is implicit in R.M. Smullyan's book *First-Order Logic*, Springer-Verlag, 1968 (see the proofs of Theorem 6 at the end of Chapter VI and Theorem 2 in Chapter VII). And it is explicit in G. Kreisel's and J.L. Krivine's book *Elements of Mathematical Logic*, 1967 (see their *Finiteness Theorem*, Theorem 12, in Chapter 2). However, it takes some doing to decode the notation in these two books, somewhat different from that in more recent publications.

An additional benefit of an excursion through *Herbrand theory* is that it has other important uses outside these notes. It plays the role of a *transfer principle* by reducing many questions of first-order logic to questions of propositional logic. In particular, it provides a unified framework for the study of other topics later in the course, such as the *tableaux* and *resolution* methods for first-order logic.

As an intermediate stage facilitating the transition from propositional logic to first-order logic, I also include (in Section 3) the reduction of Compactness for the logic of *quantified Boolean formulas* (QBF's) to Compactness for propositional logic.

1 Compactness for Propositional Logic

We say a set Γ of WFF's is *finitely satisfiable* iff every finite subset of Γ is satisfiable. If Γ is a finite set, then "finitely satisfiable" coincides with "satisfiable".

We write $\mathsf{models}(\Gamma)$ to denote the set of models of Γ. In the propositional case, $\mathsf{models}(\Gamma)$ is the set of all Boolean valuations of the propositional variables that satisfy every WFF in Γ. The next lemma is a preliminary result for the Compactness Theorem.

Lemma 1. *Let Γ be a set of propositional WFF's and φ an arbitrary propositional WFF. If Γ is finitely satisfiable, then $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$ (or possibly both) is finitely satisfiable.*

Proof. Suppose the conclusion of the lemma does not hold: Both $\Gamma\cup\{\varphi\}$ and $\Gamma\cup\{\neg\varphi\}$ are not finitely satisfiable. Hence, there are finite subsets $\Gamma_1 \subseteq \Gamma$ and $\Gamma_2 \subseteq \Gamma$ such that both $\Gamma_1 \cup \{\varphi\}$ and $\Gamma_2 \cup \{\neg\varphi\}$ are not satisfiable. Hence, both:

$$\mathsf{models}(\Gamma_1) \cap \mathsf{models}(\varphi) \;=\; \varnothing \qquad \text{and} \qquad \mathsf{models}(\Gamma_2) \cap \mathsf{models}(\neg\varphi) \;=\; \varnothing .$$

Hence, both $\mathsf{models}(\Gamma_1) \subseteq \mathsf{models}(\neg\varphi)$ and $\mathsf{models}(\Gamma_2) \subseteq \mathsf{models}(\varphi)$. Hence,

$$\mathsf{models}(\Gamma_1 \cup \Gamma_2) \;=\; \mathsf{models}(\Gamma_1) \cap \mathsf{models}(\Gamma_2) \;\subseteq\; \mathsf{models}(\neg\varphi) \cap \mathsf{models}(\varphi) \;=\; \varnothing .$$

Hence, the finite subset $\Gamma_1 \cup \Gamma_2$ does not have models, *i.e.*, is not satisfiable. Hence, Γ is not finitely satisfiable, and the hypothesis of the lemma does not hold either. □

Theorem 2 (Compactness for Propositional Logic)**.** *Let Γ be a set of propositional WFF's. Then Γ is satisfiable iff Γ is finitely satisfiable.*

Proof. The left-to-right implication is immediate. The non-trivial is the right-to-left implication: If Γ is finitely satisfiable, then Γ is satisfiable.

Let $X = \{x_1, x_2, x_3, \ldots\}$ be the set of propositional variables, finite or countably infinite. Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be a fixed, countably infinite, enumeration of all the WFF's of propositional logic over X and the logical connectives. This enumeration of the φ_i's is countably infinite. We define a nested sequence of supersets of Γ as follows:

$$\Delta_0 \quad = \Gamma,$$

$$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{\varphi_i\} & \text{if } \Delta_i \cup \{\varphi_i\} \text{ is finitely satisfiable,} \\ \Delta_i \cup \{\neg\varphi_i\} & \text{otherwise.} \end{cases}$$

Clearly, $\Gamma = \Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \Delta_3 \subseteq \cdots$. By induction on $i \geqslant 0$, using Lemma 1, every Δ_i is a finitely satisfiable set of propositional WFF's. We now define:

$$\Delta \;=\; \bigcup_i \Delta_i \qquad \text{(the limit of the } \Delta_i\text{'s)}$$

Two facts about Δ follow from its definition:

1. For every propositional WFF φ, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$, but not both.
   This is why Δ is said *maximal finitely satisfiable*, soon to be shown just *maximal satisfiable*.
2. Since every propositional variable x_i is a WFF itself, either $x_i \in \Delta$ or $\neg x_i \in \Delta$, but not both.

We next define a Boolean valuation σ as follows:

$$\sigma(x_i) \;=\; \begin{cases} \mathsf{T} & \text{if } x_i \in \Delta, \\ \mathsf{F} & \text{if } \neg x_i \in \Delta. \end{cases}$$

Claim: σ satisfies a propositional WFF φ iff $\varphi \in \Delta$. We leave the proof of this claim as an (easy) exercise.

Hence, σ is a valuation satisfying every WFF in Δ, *i.e.*, $\sigma \in \mathsf{models}(\Delta)$. Hence, because $\Gamma \subseteq \Delta$, it it also the case that σ satisfies every WFF in Γ. Hence, Γ is satisfiable. □

Exercise 3. Provide the details in the preceding proof showing that there is "a fixed, countably infinite, enumeration of all the WFF's of propositional logic over X". Although not needed in the proof, we can state a stronger assertion: The fixed enumeration of all the WFF's of propositional logic is computable, *i.e.*, can be generated by an infinitely-running computer program. □

Exercise 4. In the definition of the nested sequence of Δ_i's in the preceding proof, we did *not* write:

$$\Delta_{i+1} \;=\; \begin{cases} \Delta_i \cup \{\varphi_i\} & \text{if } \Delta_i \cup \{\varphi_i\} \text{ is finitely satisfiable,} \\ \Delta_i \cup \{\neg\varphi_i\} & \text{if } \Delta_i \cup \{\neg\varphi_i\} \text{ is finitely satisfiable.} \end{cases}$$

Explain why. *Hint*: Exhibit a set Γ of WFF's and a single WFF φ such that both $\Gamma\cup\{\varphi\}$ and $\Gamma\cup\{\neg\varphi\}$ are satisfiable. □

Exercise 5. Prove the **claim** in the proof of Theorem 2. There is no harm in simplifying the syntax a little, by restricting the logical connectives to two, say, $\{\neg, \vee\}$ or $\{\neg, \wedge\}$. *Hint*: Use structural induction on propositional WFF's. □

Lemma 6. *Let Γ be a set of propositional WFF's and φ an arbitrary propositional WFF. Then $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable – or, equivalently, $\Gamma \not\models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is satisfiable.*

Proof. We have the following sequence of equivalences:

$$\begin{aligned} \Gamma \models \varphi \quad &\text{iff} \quad \mathsf{models}(\Gamma) \subseteq \mathsf{models}(\varphi) \\ &\text{iff} \quad \mathsf{models}(\Gamma) \cap \mathsf{models}(\neg\varphi) = \varnothing \\ &\text{iff} \quad \mathsf{models}(\Gamma \cup \{\neg\varphi\}) = \varnothing \\ &\text{iff} \quad \Gamma \cup \{\neg\varphi\} \text{ is unsatisfiable,} \end{aligned}$$

which is the desired conclusion. □

Corollary 7. *Let Γ be a set of propositional WFF's and φ an arbitrary propositional WFF. Then $\Gamma \models \varphi$ iff then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.*

Proof. The right-to-left implication is immediate. For the left-to-right implication, we prove the contrapositive. So, suppose $\Gamma_0 \not\models \varphi$ for every finite subset $\Gamma_0 \subseteq \Gamma$. We have the following equivalences:

$$\begin{aligned} \Gamma_0 \not\models \varphi \text{ for every finite } \Gamma_0 \subseteq \Gamma \quad &\text{iff} \quad \Gamma_0 \cup \{\neg\varphi\} \text{ satisfiable for every finite } \Gamma_0 \subseteq \Gamma \text{ (by Lemma 6)} \\ &\text{iff} \quad \Gamma \cup \{\neg\varphi\} \text{ finitely satisfiable (by definition)} \\ &\text{iff} \quad \Gamma \cup \{\neg\varphi\} \text{ satisfiable (by Theorem 2)} \\ &\text{iff} \quad \Gamma \not\models \varphi \text{ (by Lemma 6)}\ , \end{aligned}$$

which is the desired conclusion. □

Exercise 8. We can restrict the logical connectives of propositional logic to $\{\neg, \vee, \wedge\}$. Assume the set $X = \{x_1, x_2, x_3, \ldots\}$ of propositional variables is countably infinite. Suppose we extend this syntax with two new connectives, denoted $\bigvee$ and $\bigwedge$, each taking as a single argument a countably infinite set of previously defined WFF's. The resulting syntax is one version of what is called the *infinitary propositional logic*. If Γ is a countably infinite set of the form $\Gamma = \{\varphi_1, \varphi_2, \varphi_3, \ldots\}$, then:

$$\bigvee \Gamma \;=\; \varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \cdots ,$$

and similarly for $\bigwedge \Gamma$. There are three parts in this exercise:

1. Define the syntax of the infinitary PL, preferably in an extended BNF (Backus-Naur form).
2. Define the semantics of the infinitary PL, by structural induction on the syntax in Part 1, starting from an assignment σ of truth values to every member of X (for the base case of the induction).
3. Show that Theorem 2 does not hold, and therefore nor does Corollary 7, for the infinitary PL. *Hint*: Define a countably infinite set Γ of WFF's such that every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable, but Γ is not. *Further Hint*: Include the WFF $\varphi = \bigvee\{\neg x_1, \neg x_2, \neg x_3, \ldots\}$ in your proposed Γ.

□

2 Completeness for Propositional Logic

The next lemma, which does not need Compactness for its proof, is a weaker form of the Completeness Theorem. The Completeness Theorem in full generality is Theorem 10 whose proof uses Compactness in an essential way (in the form of Corollary 7).

Lemma 9. *Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional WFF's. If $\varphi_1, \ldots, \varphi_n \models \psi$ then $\varphi_1, \ldots, \varphi_n \vdash \psi$.*

Proof. This lemma is the Completeness Theorem as stated in the book [LCS], in Section 1.4.4; specifically, this is the left-to-right implication in Corollary 1.39.[5] Section 1.4.4 in [LCS] is entirely devoted to the details of the proof. □

The details of the preceding proof very much depend on the kind of proof system it is based on. So, in the book [LCS], it depends on the system of *natural deduction*. But Lemma 9, or lemmas essentially asserting the same thing, in fact hold again for all the finitary proof systems of propositional logic, *e.g.*, all the systems surveyed in earlier handouts (click here). The phrase "finitary proof system" here is a bit loose, but you can take it to qualify a formal system that generates new finite expressions (*e.g.*, the sequents of propositional logic in natural-deduction style or the WFF's of propositional logic in Hilbert style) from previously generated ones by means of finitely many rules that require each finitely many antecedents – without using any notion of "limit", any notion of infinite sequence in its formulation, and any notion of infinite set.[6]

Theorem 10 (Completeness for Propositional Logic)**.** *Let Γ be a set of propositional WFF's (possibly infinite), and ψ a propositional WFF. If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.*

Proof. If $\Gamma \models \psi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \psi$, by Corollary 7. By Lemma 9, it follows that $\Gamma_0 \vdash \psi$. Padding Γ_0 with the redundant premises in $(\Gamma - \Gamma_0)$, we conclude $\Gamma \vdash \psi$. □

[5] Michael Huth and Mark Ryan, *Logic in Computer Science*, Second Edition, Cambridge University Press, 2004.

[6] If you want to go deeper into what "finitary proofs" and "finitary proof systems" mean, search the Web for these two expressions. Also search the Web for notions of "limit" in mathematics and how they are used.

3 Compactness and Completeness for the Logic of QBF's

We can prove Compactness for the logic of *quantified Boolean formulas* (QBF's) in one of two ways: *either* directly *or* by reducing it to Compactness for propositional logic. We choose the latter approach in these notes because we have already done much of the preliminary work in Section 1 and as an intermediate stage before studying Compactness for first-order logic in Section 5.

Lemma 11. *Let* Γ *be a set (finite or infinite) of QBF's. We can construct a set* Γ' *of propositional WFF's such that:*

1. Γ *is finitely satisfiable iff* Γ' *is finitely satisfiable.*
2. Γ *is satisfiable iff* Γ' *is satisfiable.*

The construction in the proof below establishes a stronger result: Γ and Γ' are more than *finitely equisatisfiable* and *equisatisfiable*; they are in fact *equivalent*. Specifically, for every QBF $\varphi \in \Gamma$ there is a propositional WFF $\varphi' \in \Gamma'$ such that φ and φ' are equivalent; and, similarly, for every propositional WFF $\varphi' \in \Gamma'$ there is a QBF $\varphi \in \Gamma$ such that φ and φ' are equivalent.

Proof. If φ is a propositional WFF, we write "$\varphi[x := \bot]$" and "$\varphi[x := \top]$" to denote the substitution of $\bot$ and $\top$, respectively, for every occurrence of variable x in φ.

We define a transformation $\boldsymbol{\Theta}(\ )$ from QBF's to propositional WFF's by structural induction:

1. $\boldsymbol{\Theta}(x) \triangleq x$ (for every propositional variable x)
2. $\boldsymbol{\Theta}(\neg\varphi) \triangleq \neg\boldsymbol{\Theta}(\varphi)$
3. $\boldsymbol{\Theta}(\varphi \wedge \psi) \triangleq \boldsymbol{\Theta}(\varphi) \wedge \boldsymbol{\Theta}(\psi)$
4. $\boldsymbol{\Theta}(\varphi \vee \psi) \triangleq \boldsymbol{\Theta}(\varphi) \vee \boldsymbol{\Theta}(\psi)$
5. $\boldsymbol{\Theta}(\varphi \to \psi) \triangleq \boldsymbol{\Theta}(\varphi) \to \boldsymbol{\Theta}(\psi)$
6. $\boldsymbol{\Theta}(\forall x\ \varphi) \triangleq \boldsymbol{\Theta}(\varphi)[x := \bot]\ \wedge\ \boldsymbol{\Theta}(\varphi)[x := \top]$ (for every occurrence of x in $\boldsymbol{\Theta}(\varphi)$)
7. $\boldsymbol{\Theta}(\exists x\ \varphi) \triangleq \boldsymbol{\Theta}(\varphi)[x := \bot]\ \vee\ \boldsymbol{\Theta}(\varphi)[x := \top]$ (for every occurrence of x in $\boldsymbol{\Theta}(\varphi)$)

Claim: For every QBF φ, the transformation $\boldsymbol{\Theta}(\varphi)$ satisfies the following properties:

(a) $\boldsymbol{\Theta}(\varphi)$ is a propositional WFF,

(b) $\mathsf{FV}(\varphi)$ are exactly all the propositional variables occurring in $\boldsymbol{\Theta}(\varphi)$, and

(c) if $X = \mathsf{FV}(\varphi)$, then for every assignment σ of truth values to the members of X, it holds that σ satisfies φ iff σ satisfies $\boldsymbol{\Theta}(\varphi)$.

Part (c) in this claim shows that φ and $\boldsymbol{\Theta}(\varphi)$ are not only equisatisfiable, but also equivalent. We leave the proof of this claim as an exercise. Given an arbitrary (finite or infinite) set Γ of QBF's, we now define Γ' by:

$$\Gamma' \triangleq \left\{ \boldsymbol{\Theta}(\varphi) \;\middle|\; \varphi \in \Gamma \right\}$$

By the preceding claim, we conclude that Γ' is a set of propositional WFF's, defined over the set of variables $X = \mathsf{FV}(\Gamma)$, such that for every truth-value assignment σ for the variables in X:

- for every finite subset $\Delta \subseteq \Gamma$ there is a finite subset $\Delta' \subseteq \Gamma'$ s.t. σ satisfies Δ iff σ satisfies Δ',

- for every finite subset $\Delta' \subseteq \Gamma'$ there is a finite subset $\Delta \subseteq \Gamma$ s.t. σ satisfies Δ iff σ satisfies Δ',
- σ satisfies Γ iff σ satisfies Γ'.

We leave the missing details in the proof of the preceding three bullet points as an exercise. □

Exercise 12. Prove the **claim** in the proof of Lemma 11. *Hint*: Use structural induction on QBF's, following the seven steps in the definition of the transformation $\boldsymbol{\Theta}(\ )$. □

Exercise 13. In the statement of Lemma 11 and its proof, the set Γ of QBF's and the set Γ' of propositional WFF's are equivalent. Specify:

1. Conditions under which $|\Gamma| = |\Gamma'|$, and
2. Conditions under which $|\Gamma| > |\Gamma'|$,

where $|\Gamma|$ is the cardinality of the set Γ. *Hint*: Consider, for example, the case when all the QBF's in Γ are *closed*; what is Γ' in this case? □

Exercise 14. Supply the missing details in the proof of the three bullet points at the end of the proof of Lemma 11. *Hint*: This is subtler than at first blush; do Exercise 13 before you attempt this one. □

Theorem 15 (Compactness for the Logic of QBF's)**.** *Let Γ be a set of QBF's. Then Γ is satisfiable iff Γ is finitely satisfiable.*

Proof. The left-to-right implication is immediate. The non-trivial is the right-to-left implication, *i.e.*, we have to prove that if Γ is finitely satisfiable, then Γ is satisfiable. Let Γ' be the set of propositional WFF's defined from Γ according to Lemma 11.

By Lemma 11, Γ is finitely satisfiable iff Γ' is finitely satisfiable. By Theorem 2, Γ' is finitely satisfiable iff Γ' is satisfiable. By Lemma 11 once more, Γ' is satisfiable iff Γ is satisfiable. Hence, if Γ is finitely satisfiable, then Γ is satisfiable, as desired. □

For the next lemma and its corollary, review the formal semantics of QBF's in Handout 13. The meaning of "$\models$" for QBF's depends on this formal semantics.

Lemma 16. *Let Γ be a set of QBF's and φ an arbitrary QBF. Then $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable – or, equivalently, $\Gamma \not\models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is satisfiable.*

Proof. Identical to the proof of Lemma 6, except that here Γ is a set of QBF's and φ is a QBF. □

Corollary 17. *Let Γ be a set of QBF's and φ an arbitrary QBF. Then $\Gamma \models \varphi$ iff then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.*

Proof. Identical to the proof of Corollary 7, except that here Γ is a set of QBF's and φ is a QBF. Moreover, here we invoke Lemma 16 instead of Lemma 6, and Theorem 15 instead of Theorem 2. □

Before turning to Completeness for the logic of QBF's, review the proof rules for QBF's in Handout 13. Although the proof rules in Handout 13 are in *natural deduction* style, Completeness holds for any of the other available proof systems for the logic of QBF's.

The next lemma is a weaker form of the Completeness Theorem for QBF's, *i.e.*, it is restricted to a finite set of formulas $\{\varphi_1, \ldots, \varphi_n\}$. The Completeness Theorem for QBF's in full generality is Theorem 20.

Lemma 18. *Let $\varphi_1, \ldots, \varphi_n, \psi$ be QBF's. If $\varphi_1, \ldots, \varphi_n \models \psi$ then $\varphi_1, \ldots, \varphi_n \vdash \psi$.*

Proof. This lemma is proved in the same way as Lemma 9, following the steps of the proof of the Completeness Theorem for propositional logic, as stated in the book [LCS], in Section 1.4.4; specifically, this is the left-to-right implication in Corollary 1.39.[7] □

Exercise 19. Write the details of the proof for Lemma 18. □

Theorem 20 (Completeness for the Logic of QBF's)**.** *Let* Γ *be a set of QBF's (possibly infinite), and* ψ *a QBF. If* $\Gamma \models \psi$*, then* $\Gamma \vdash \psi$*.*

Proof. Identical to the proof of Theorem 10, except that all formulas are now QBF's, not just propositional WFF's. Moreover, we need to invoke Corollary 17 instead of Corollary 7, and Lemma 18 instead of Lemma 9. □

4 Herbrand Theory

To prove Compactness for first-order logic, we want to use once more a reduction to Compactness for propositional logic, but the situation is a little more complicated than it was for QBF's in Section 3. What we need is a lemma that is the counterpart of Lemma 11 extended to first-order WFF's. This requires some extra preliminary work, represented by what is often called *Herbrand theory.*[8]

A first-order WFF φ is in *prenex normal form* iff φ consists of a (possibly empty) string of quantifiers followed by a quantifier-free WFF. The string of quantifiers is the *prefix* of φ, and the quantifier-free WFF is the *matrix* of φ. This definition applies to both QBF's and first-order WFF's.

In Handout 13 on QBF's, there is an inductive definition for the transformation of an arbitrary QBF into an equivalent QBF in prenex normal form. Call this transformation $\boldsymbol{\Theta}_{\text{pr}}(\,)$. We can apply $\boldsymbol{\Theta}_{\text{pr}}(\,)$ just as defined in Handout 13 to an arbitrary first-order WFF φ in order to obtain an equivalent first-order WFF $\varphi' = \boldsymbol{\Theta}_{\text{pr}}(\varphi)$ in prenex normal form.

Given a first-order WFF φ in prenex normal form, the *Skolemization* of φ is a first-order WFF ψ obtained by initially setting ψ to φ and then by repeatedly applying a three-step sequence to ψ:[9]

1. Find the leftmost $\exists$ in the quantifier prefix of ψ, which binds a variable x and appears as "$\exists\, x$",
2. Introduce a fresh function symbol f_x of arity equal to the number of $\forall$'s to the left of "$\exists\, x$",
3. If the $\forall$'s to the left of "$\exists\, x$" are "$\forall y_1 \cdots \forall y_n$", then cross out "$\exists\, x$" from the quantifier prefix and replace all occurrences of x in the matrix of ψ by the term $f_x(y_1, \ldots, y_n)$.

Applying these three steps once eliminates one existential quantifier from the quantifier prefix, and applying them repeatedly eliminates all the existential quantifiers, finitely many of them. The final WFF ψ is therefore in prenex normal form where the quantifier prefix consists of $\forall$'s only; *i.e.*, Skolemizing a prenex normal form φ produces a *universal* WFF ψ.

Every time the three-step sequence is applied, a fresh function symbol f_x is introduced. There are as many new fresh function symbols as there are existential quantifiers in the prefix of the initial WFF φ in prenex normal form. These fresh function symbols are called *Skolem functions.* Note that if the leftmost "$\exists\, x$" in the initial φ is not preceded by any $\forall$, the associated Skolem function f_x has arity $= 0$, *i.e.*, f_x is a constant symbol.

[7] Michael Huth and Mark Ryan, *Logic in Computer Science*, Second Edition, Cambridge University Press, 2004.
[8] Jacques Herbrand is a mathematician of the early twentieth century who laid out the foundation for this theory.
[9] The words *Skolemize* and *Skolemization* are derived from the name of the mathematical logician Thoralf Skolem. If you want to find out more about the many uses of *Skolemization*, click here.

We write $\boldsymbol{\Theta}_{\rm sk}(\varphi)$ to denote the Skolemization of the WFF φ in prenex normal form. If φ is an arbitrary first-order WFF, not necessarily in prenex normal form, then we write $\boldsymbol{\Theta}_{\rm pr,sk}(\varphi)$ to denote the two-stage transformation of φ – first, into prenex normal form and, second, into Skolemized form – and we also call $\boldsymbol{\Theta}_{\rm pr,sk}(\varphi)$ the Skolemization of φ.

While φ and $\boldsymbol{\Theta}_{\rm pr}(\varphi)$ are logically equivalent, it does not make sense to talk about the equivalence (or non-equivalence) of φ and $\boldsymbol{\Theta}_{\rm pr,sk}(\varphi)$ because the signature of the latter WFF is different from the signature of φ. Nevertheless, we have the following result. Recall that a *sentence* φ is a closed formula, *i.e.*, $\mathsf{FV}(\varphi) = \varnothing$.

Lemma 21. *Let φ be an arbitrary first-order sentence. Then φ is satisfiable iff $\boldsymbol{\Theta}_{\rm pr,sk}(\varphi)$ is satisfiable.*

Proof. We can assume that φ is already in prenex normal form. It suffices to show how the elimination of the leftmost existential quantifier from the prefix of φ produces another prenex normal form ψ which is equisatisfiable with φ, and then the same process can be repeated for the elimination of all the other existential quantifers in the prefix of φ. Let then φ be of the form:

$$\varphi \triangleq \forall x_1 \cdots \forall x_n \exists y\ \varphi_0$$

where $n \geqslant 0$ and φ_0 is a a prenex normal form such that $\mathsf{FV}(\varphi_0) \subseteq \mathsf{FV}(\varphi) \cup \{x_1, \ldots, x_n, y\}$. According to the Skolemization process, ψ is of the form:

$$\psi \triangleq \forall x_1 \cdots \forall x_n\ \varphi_0[y := f_y(x_1, \ldots, x_n)]$$

where f_y is a fresh n-ary function symbol. Σ and $\Sigma \cup \{f_y\}$ are the signatures of φ and ψ, respectively.

Let $\mathcal{M}$ be a structure for signature Σ. The expansion $\mathcal{M}' \triangleq (\mathcal{M}, f_y^{\mathcal{M}'})$ of $\mathcal{M}$ is a structure for signature $\Sigma \cup \{f_y\}$. Let A be the universe of $\mathcal{M}$, which is also the universe of $\mathcal{M}'$. If $\mathcal{M}' \models \psi$, then clearly $\mathcal{M} \models \varphi$. Hence, if ψ is satisfiable, then so is φ.

Conversely, let $\mathcal{M} \models \varphi$. We construct a structure $\mathcal{M}'$ for $\Sigma \cup \{f_y\}$ by expanding $\mathcal{M}$ so that for every $a_1, \ldots, a_n \in A$, the function $f_y^{\mathcal{M}'}$ maps $(a_1, \ldots, a_n)$ to b where $\mathcal{M}, a_1, \ldots, a_n, b \models \varphi_0$. Hence, $\mathcal{M}' \models \psi$. Hence, if φ is satisfiable, then so is ψ. □

Exercise 22. What goes wrong in the proof of Lemma 21 if φ is an open WFF? *Hint*: Try the open WFF $\varphi(y) \triangleq \exists! v \forall w \big(P(a, w) \wedge P(v, w)\big) \to \exists x \big(P(a, y) \wedge P(x, y)\big)$, where "$\exists!$" means "there exists exactly one", P is a binary predicate symbol and a is a constant symbol. Show that $\models \varphi(y)$, but the construction in the proof of Lemma 21 produces an open WFF $\psi(y)$ not satisfied by any structure $\mathcal{M}$ – unless we introduce additional constraints at the meta-level on $\mathcal{M}$. □

Exercise 23. Let P be a binary predicate symbol and f a unary function symbol.

1. Show that the sentence $\varphi \triangleq \forall x\, P\big(x, f(x)\big) \to \forall x \exists y\, P(x, y)$ is valid, *i.e.*, formally provable. Do it in two different ways:
   (a) proof-theoretically, $\vdash \varphi$, using natural deduction, and
   (b) semantically, $\models \varphi$.
2. Show that the sentence $\psi \triangleq \forall x \exists y\, P(x, y) \to \forall x\, P\big(x, f(x)\big)$ is not valid, *i.e.*, not formally provable. Note that ψ is just the converse implication of φ.

   Hint: Try a semantic approach, *i.e.*, show $\not\models \psi$. You need to define a structure $\mathcal{M}$ so that the left-hand side of "$\to$" in ψ is true in $\mathcal{M}$ but the right-hand side of "$\to$" is false in $\mathcal{M}$.
3. Conclude that $\forall x \exists y\, P(x, y)$ and $\forall x\, P\big(x, f(x)\big)$ are not equivalent first-order WFF's.

Remark: Despite the conclusion in part 3, Lemma 21 asserts that $\forall x \exists y\, P(x, y)$ and $\forall x\, P\big(x, f(x)\big)$ are equisatisfiable, *i.e.*, if there is a model for one, then there is a model for the other, and vice-versa. □

Herbrand universes, Herbrand structures, Herbrand models

Consider a fixed first-order signature $\Sigma = \{\mathcal{P}, \mathcal{F}, \mathcal{C}\}$, where $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$, are countable sets of predicate symbols, function symbols, and constant symbols, respectively. If Γ is a set of WFF's over Σ, we write $\mathcal{P}(\Gamma)$, $\mathcal{F}(\Gamma)$, and $\mathcal{C}(\Gamma)$, to respectively denote the sets of predicate symbols, function symbols, and constant symbols, occurring in Γ. We write $\Sigma(\Gamma)$ to denote $\{\mathcal{P}(\Gamma), \mathcal{F}(\Gamma), \mathcal{C}(\Gamma)\}$.

If $\mathcal{C}(\Gamma) = \varnothing$, we add a fresh constant symbol to it in order to be able to build a non-empty set of variable-free terms. The variable-free terms and the variable-free atomic formulas over $\Sigma(\Gamma)$ are called the *ground terms* and the *ground atoms* over $\Sigma(\Gamma)$, respectively. More precisely, $\mathsf{Gr\_Terms}(\Gamma)$ is the least set satisfying the condition:

$$\begin{aligned}\mathsf{Gr\_Terms}(\Gamma) \;\supseteq\; & \mathcal{C}(\Gamma) \cup \\ & \{\, f(t_1, \ldots, t_n) \;\big|\; f \in \mathcal{F}(\Gamma) \text{ has arity } n \geqslant 1,\; t_1, \ldots, t_n \in \mathsf{Gr\_Terms}(\Gamma) \,\},\end{aligned}$$

and $\mathsf{Gr\_Atoms}(\Gamma)$ is the set defined by:

$$\begin{aligned}\mathsf{Gr\_Atoms}(\Gamma) \;\triangleq\; & \{\, t_1 \doteq t_2 \mid t_1, t_2 \in \mathsf{Gr\_Terms}(\Gamma) \,\} \cup \\ & \{\, P(t_1, \ldots, t_n) \mid P \in \mathcal{P}(\Gamma) \text{ has arity } n \geqslant 0,\; t_1, \ldots, t_n \in \mathsf{Gr\_Terms}(\Gamma) \,\}.\end{aligned}$$

If $\mathcal{F}(\Gamma)$ contains function symbols of arity $\geqslant 1$, then $\mathsf{Gr\_Terms}(\Gamma)$ is countably infinite, otherwise it is a non-empty finite set (because $\mathcal{C}(\Gamma) \neq \varnothing$).

The *Herbrand universe* over $\Sigma(\Gamma)$, *i.e.*, induced by the signature of Γ, is the set $\mathsf{Gr\_Terms}(\Gamma)$. A *Herbrand structure* $\mathcal{H}(\Gamma)$ over signature $\Sigma(\Gamma)$ – or just denoted $\mathcal{H}$ if Γ is understood from the context – is a structure whose universe is $\mathsf{Gr\_Terms}(\Gamma)$:

$$\mathcal{H} \;\triangleq\; \big(\mathsf{Gr\_Terms}(\Gamma), \mathcal{P}(\Gamma)^{\mathcal{H}}, \mathcal{F}(\Gamma)^{\mathcal{H}}, \mathcal{C}(\Gamma)^{\mathcal{H}}\big)$$

where $\mathcal{P}(\Gamma)^{\mathcal{H}}$ denotes the interpretation of every predicate symbol in $\mathcal{P}(\Gamma)$ in $\mathcal{H}$, and similarly for $\mathcal{F}(\Gamma)^{\mathcal{H}}$ and $\mathcal{C}(\Gamma)^{\mathcal{H}}$.

We need to handle a variable-free expression such as "$f(a, b)$" with care, because it is both an (uninterpreted) term and an element of the universe $\mathsf{Gr\_Terms}(\Gamma)$. The context will disambiguate the sense in which "$f(a, b)$" is understood.[10]

The interpretation $t^{\mathcal{H}}$ of a ground term t is the term t itself. The interpretation $f^{\mathcal{H}}$ of a function symbol f of arity $n \geqslant 1$ is given by $f^{\mathcal{H}}(t_1, \ldots, t_n) \triangleq f(t_1, \ldots, t_n)$ for all $t_1, \ldots, t_n \in \mathsf{Gr\_Terms}(\Gamma)$. Observe that *the interpretation of terms and function symbols is identical in all Herbrand structures over the same signature* $\Sigma(\Gamma)$.

Only the interpretation of predicate symbols differs from one Herbrand structure to another over the same $\Sigma(\Gamma)$. A Herbrand structure is therefore uniquely determined by the truth values assigned to the members of $\mathsf{Gr\_Atoms}(\Gamma)$, which is thus sometimes called the *Herbrand base* or *ground base*.

As usual, the interpretation of an arbitrary WFF φ (containing free variables in general) in a Herbrand structure $\mathcal{H}$ over signature $\Sigma(\Gamma)$ is relative to an environment $\ell : \mathcal{V} \to \mathsf{Gr\_Terms}(\Gamma)$ where $\mathcal{V}$ is the set of first-order variables and $\mathsf{Gr\_Terms}(\Gamma)$ is the universe of $\mathcal{H}$. We say Γ has a *Herbrand model* iff the Herbrand structure $\mathcal{H}$ induced by the signature of Γ satisfies every $\varphi \in \Gamma$, *i.e.*, iff:

$$\Gamma \;=\; \Big\{ \varphi \in \Gamma \;\Big|\; \mathcal{H} \models \varphi \Big\}.$$

If Γ is the singleton set $\{\varphi\}$, we write $\mathcal{P}(\varphi)$, $\mathcal{F}(\varphi)$, $\mathcal{C}(\varphi)$, $\mathsf{Gr\_Terms}(\varphi)$, etc., instead of $\mathcal{P}(\{\varphi\})$, $\mathcal{F}(\{\varphi\})$, $\mathcal{C}(\{\varphi\})$, $\mathsf{Gr\_Terms}(\{\varphi\})$, etc.

[10] Other names for a Herbrand structure in the literature are *canonical structure* and *(close) term structure*.

Lemma 24. *Let φ be an arbitrary first-order sentence. Then φ is satisfiable iff $\boldsymbol{\Theta}_{\mathrm{pr,sk}}(\varphi)$ has a Herbrand model.*

Proof. Let $\psi \triangleq \boldsymbol{\Theta}_{\mathrm{pr,sk}}(\varphi)$. If ψ has a model, Herbrand or not, then ψ is satisfiable. By Lemma 21, if ψ is satisfiable, then φ is satisfiable. The converse is more delicate to prove.

Suppose φ is satisfiable. By Lemma 21 again, ψ is satisfiable. Hence, there is a structure $\mathcal{M}$ with signature $\Sigma(\psi)$ satisfying ψ, *i.e.*, $\mathcal{M} \models \psi$. We need to show there is a Herbrand structure satisfying ψ, *i.e.*, ψ has a Herbrand model. We proceed by first specifying a Herbrand structure $\mathcal{H}$ induced by ψ, and then by showing that $\mathcal{H}$ satisfies ψ. By definition, the signature of $\mathcal{H}$ is $\Sigma(\psi)$, which is also that of $\mathcal{M}$. By definition again, the universe of $\mathcal{H}$ and the interpretations of every $f \in \mathcal{F}(\psi)$ and every $c \in \mathcal{C}(\psi)$ are already fixed, namely:

- the universe of $\mathcal{H}$ is $\mathsf{Gr\_Terms}(\psi)$,
- $f^{\mathcal{H}}(t_1, \ldots, t_n) \triangleq f(t_1, \ldots, t_n)$ for every n-ary $f \in \mathcal{F}(\psi)$ and $t_1, \ldots, t_n \in \mathsf{Gr\_Terms}(\psi)$,
- $c^{\mathcal{H}} \triangleq c$ for every $c \in \mathcal{C}(\psi)$.

Only the interpretation of the predicate symbols need to be specified, which we set as follows:

- $(t_1, \ldots, t_n) \in P^{\mathcal{H}}$ iff $(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}) \in P^{\mathcal{M}}$
  for every n-ary $P \in \mathcal{P}(\psi)$ and $t_1, \ldots, t_n \in \mathsf{Gr\_Terms}(\psi)$.

In the last bullet point, we do *not* include "$\doteq$" as a binary predicate in $\mathcal{P}(\psi)$. Hence, it is not the case that $(t_1 \doteq t_2)$ holds in $\mathcal{H}$ iff $(t_1^{\mathcal{M}} \doteq t_2^{\mathcal{M}})$ holds in $\mathcal{M}$. However, if $(t_1 \doteq t_2)$ holds in $\mathcal{H}$, *i.e.*, t_1 and t_2 are the same expression, then necessarily $(t_1^{\mathcal{M}} \doteq t_2^{\mathcal{M}})$ in $\mathcal{M}$. To conclude the proof, we prove a stronger assertion, namely: *For every sentence α in Skolem form such that $\Sigma(\alpha) \subseteq \Sigma(\psi)$, it holds that $\mathcal{M} \models \alpha$ iff $\mathcal{H} \models \alpha$.*, which we prove by induction on the number $k \geqslant 0$ of universal quantifiers in α.

1. *Basis step*: $k = 0$, in which case α has no quantifiers, *i.e.*, α is a propositional combination of elements in $\mathsf{Gr\_Atoms}(\alpha)$. For this basis step, we proceed by induction on the number of propositional connectives in α, which can be limited to $\{\neg, \wedge\}$. Remaining details of this induction are straightforward and left to you.
2. *Induction hypothesis*: The assertion holds for every sentence α in Skolem form with k universal quantifiers, for some $k \geqslant 0$.
3. *Induction step*: Let $\beta \triangleq \forall x\, \alpha(x)$ be an arbitrary Skolem form where $\alpha(x)$ has one free variable x and $k \geqslant 0$ universal quantifiers, and β has $k + 1$ universal quantifiers.

We prove the *induction step* by a sequence of equivalences. Let U be the universe of $\mathcal{M}$. We write "$[x \mapsto u]$" to denote the part of an environment that maps the free variable x to the element $u \in U$:

$\mathcal{M} \models \forall x\, \alpha(x)$

iff for all $u \in U$, it holds that $\mathcal{M}, [x \mapsto u] \models \alpha$

iff for all $u \in U$ of the form $u = t^{\mathcal{M}}$ where $t \in \mathsf{Gr\_Terms}(\psi)$, it holds that $\mathcal{M}, [x \mapsto u] \models \alpha$

iff for all $t \in \mathsf{Gr\_Terms}(\psi)$, it holds that $\mathcal{M}, [x \mapsto t^{\mathcal{M}}] \models \alpha$

iff for all $t \in \mathsf{Gr\_Terms}(\psi)$, it holds that $\mathcal{M} \models \alpha[x := t]$

iff for all $t \in \mathsf{Gr\_Terms}(\psi)$, it holds that $\mathcal{H} \models \alpha[x := t]$ (by the *induction hypothesis*)

iff for all $t \in \mathsf{Gr\_Terms}(\psi)$, it holds that $\mathcal{H}, [x \mapsto t^{\mathcal{H}}] \models \alpha$

iff for all $t \in \mathsf{Gr\_Terms}(\psi)$, it holds that $\mathcal{H}, [x \mapsto t] \models \alpha$ ($\mathcal{H}$ is a Herbrand structure)

iff $\mathcal{H} \models \forall x\, \alpha$

This completes the induction and the proof of the lemma. □

Exercise 25. What goes wrong in the proof of Lemma 24 if φ (and therefore ψ too) contains free variables? *Hint*: Try Exercise 22 first. And what goes wrong if ψ is not in Skolem form? *Hint*: Consider the sentence $\varphi \triangleq P(a) \wedge \exists x \neg P(x)$ which is not in Skolem form, where P is a unary predicate symbol and a is a constant symbol. Show there is a structure $\mathcal{M}$ satisfying φ but that $\mathcal{M}$ cannot be a Herbrand structure. □

Example 26. Consider the first-order WFF φ below, with signature $\Sigma(\varphi) = \big(\{P\}, \{d\}, \{a, b\}\big)$ where P is a binary predicate symbol and d is a unary function symbol:

$$\varphi \;\triangleq\; \big(a \doteq d(a)\big) \;\wedge\; \forall x \,\Big(\neg(x \doteq a) \wedge \neg(x \doteq b) \to \exists y \,\big(P(x, y) \wedge P\big(y, d(x)\big)\big)\Big)$$

φ is a sentence, since $\mathsf{FV}(\varphi) = \varnothing$. Transforming φ into prenex normal form, $\varphi_1 \triangleq \boldsymbol{\Theta}_{\text{pr}}(\varphi)$, we obtain:

$$\varphi_1 \;\triangleq\; \forall x\; \exists y \;.\; \big(a \doteq d(a)\big) \;\wedge\; \Big(\neg(x \doteq a) \wedge \neg(x \doteq b) \to \big(P(x, y) \wedge P\big(y, d(x)\big)\big)\Big)$$

Transforming the matrix of φ_1 into CNF, we obtain:

$$\varphi_2 \;\triangleq\; \forall x\; \exists y \;.\; \big(a \doteq d(a)\big) \;\wedge\; \Big((x \doteq a) \vee (x \doteq b) \vee P(x, y)\Big) \wedge \Big((x \doteq a) \vee (x \doteq b) \vee P(y, d(x))\Big)$$

Transforming φ_2 into Skolem form, $\varphi_3 \triangleq \boldsymbol{\Theta}_{\text{sk}}(\varphi_2)$, we obtain:

$$\varphi_3 \;\triangleq\; \forall x \;.\; \big(a \doteq d(a)\big) \;\wedge\; \Big((x \doteq a) \vee (x \doteq b) \vee P(x, f(x))\Big) \wedge \Big((x \doteq a) \vee (x \doteq b) \vee P(f(x), d(x))\Big)$$

where f is the Skolem function corresponding to the elimination of the existential quantifier "$\exists y$" from the prefix of φ_2. By our earlier analysis, φ_1 and φ_2 are both logically equivalent to φ, whereas φ_3 is only equisatisfiable with φ. By definition, $\mathsf{Gr\_Terms}(\varphi_3)$ is the least set of terms such that:

$$\mathsf{Gr\_Terms}(\varphi_3) \supseteq \{a, b\} \cup \Big\{\, d(t) \;\Big|\; t \in \mathsf{Gr\_Terms}(\varphi_3) \Big\} \cup \Big\{\, f(t) \;\Big|\; t \in \mathsf{Gr\_Terms}(\varphi_3) \Big\}$$

In words, every term in $\mathsf{Gr\_Terms}(\varphi_3)$ is of the form $\gamma(a)$ or $\gamma(b)$ where γ is a string of unary functions in $\{d, f\}^*$. We also have:

$$\mathsf{Gr\_Atoms}(\varphi_3) \triangleq \Big\{\, t_1 \doteq t_2 \;\Big|\; t_1, t_2 \in \mathsf{Gr\_Terms}(\varphi_3) \,\Big\} \;\cup\; \Big\{\, P(t_1, t_2) \;\Big|\; t_1, t_2 \in \mathsf{Gr\_Terms}(\varphi_3) \,\Big\}.$$

The signature of φ_3 is $\Sigma(\varphi_3) = \big(\{P\}, \{d, f\}, \{a, b\}\big)$. A Herbrand stucture induced by φ_3 is therefore of the form:

$$\mathcal{H} \;\triangleq\; \big(\mathsf{Gr\_Terms}(\varphi_3), P^{\mathcal{H}}, d^{\mathcal{H}}, f^{\mathcal{H}}, a^{\mathcal{H}}, b^{\mathcal{H}}\big) \;=\; \Big(\Big\{\gamma(\#) \;\Big|\; \# \in \{a, b\} \;\&\; \gamma \in \{d, f\}^*\Big\}, P^{\mathcal{H}}, d, f, a, b\Big),$$

where only the binary predicate $P^{\mathcal{H}}$ needs to be specified further. That is, to complete the definition of $\mathcal{H}$, we only need to assign a truth value to every member of $\mathsf{Gr\_Atoms}(\varphi_3)$. □

Exercise 27. Consider the first-order sentences φ and φ_3 in Example 26. We can take $\varphi_3 = \boldsymbol{\Theta}_{\text{pr,sk}}(\varphi)$ after transforming its quantifier-free matrix into CNF.

1. Define a structure $\mathcal{M} \triangleq (\mathbb{R}, \ldots)$ whose universe is the set $\mathbb{R}$ of all real numbers such that:
   - $\mathcal{M}$ is a model of φ,
   - $d^{\mathcal{M}}$ is *not* the identity function on $\mathbb{R}$,
   - $P^{\mathcal{M}}$ is *not* the equality relation on $\mathbb{R}$.

(We require the conditions in the second and third bullet points in order to make the exercise a little more interesting.) *Hint*: Try $P^{\mathcal{M}}$ as the usual linear ordering on $\mathbb{R}$.

2. Use the proof of Lemma 21 to define a structure $\mathcal{M}' \triangleq (\mathbb{R}, \ldots)$ such that $\mathcal{M}' \models \varphi_3$.
3. Use the proof of Lemma 24 to define a Herbrand structure $\mathcal{H} \triangleq \big(\mathsf{Gr\_Terms}(\varphi_3), \ldots\big)$ such that $\mathcal{H} \models \varphi_3$.

Note that while the initial structure $\mathcal{M}$ satisfying φ is uncountable (the cardinality of its universe $\mathbb{R}$ is uncountable), the Herbrand structure $\mathcal{H}$ satisfying φ_3 is only countably infinite (why?). ☐

Herbrand Expansions

Let φ be a first-order sentence in Skolem form, $\varphi \triangleq \forall x_1 \cdots \forall x_n.\varphi_0$, where φ_0 is the quantifier-free matrix and $\mathsf{FV}(\varphi_0) \subseteq \{x_1, \ldots, x_n\}$. The *Herbrand expansion*, also called *ground expansion*, of φ is:

$$\mathsf{Gr\_Expansion}(\varphi) \;\triangleq\; \Big\{ \varphi_0[x_1 := t_1] \cdots [x_n := t_n] \;\Big|\; t_1, \ldots, t_n \in \mathsf{Gr\_Terms}(\varphi) \Big\}.$$

In words, the set $\mathsf{Gr\_Expansion}(\varphi)$ is obtained by deleting all the universal quantifiers and replacing the variables by atomic terms in all possible ways. While φ is a single sentence, $\mathsf{Gr\_Expansion}(\varphi)$ is a set of quantifier-free sentences, and it is also infinite if $\mathsf{Gr\_Terms}(\varphi) = \mathsf{Gr\_Terms}(\varphi_0)$ is infinite.

Lemma 28. *Let φ be an arbitrary first-order sentence. Then φ is satisfiable iff the Herbrand expansion* $\mathsf{Gr\_Expansion}\big(\boldsymbol{\Theta}_{\mathrm{pr,sk}}(\varphi)\big)$ *is satisfiable.*

Proof. Straightforward consequence of Lemma 24, according to which: φ is satisfiable iff $\psi = \boldsymbol{\Theta}_{\mathrm{pr,sk}}(\varphi)$ has a Herbrand model. The universe of the Herbrand structure $\mathcal{H}$ is $\mathsf{Gr\_Terms}(\psi)$. Deletion of the universal quantifiers corresponds to replacing the variables in ψ by elements of the universe $\mathsf{Gr\_Terms}(\psi)$ in all possible ways. All formal details omitted. ☐

Exercise 29. Supply the missing formal details in the proof of Lemma 28. ☐

Let Γ be a set of quantifier-free sentences. Every sentence in Γ is therefore a propositional combination of members of $\mathsf{Gr\_Atoms}(\Gamma)$. We define a transformation $\mathcal{X}$ of every such Γ by:

$$\mathcal{X}(\Gamma) \;\triangleq\; \Big\{ \varphi[\alpha_1 := X_{\alpha_1}, \ldots, \alpha_n := X_{\alpha_n}] \;\Big|\; \varphi \in \Gamma \text{ and } \{\alpha_1, \ldots, \alpha_n\} = \mathsf{Gr\_Atoms}(\varphi) \Big\}.$$

In words, $\mathcal{X}$ replaces every ground atom α in Γ by a propositional variable X_α. (We use the upper case "X" to distinguish these propositional variables from the first-order variables written with the lower case "x".) Every expression in $\mathcal{X}(\Gamma)$ is therefore a propositional WFF over the following set of propositional variables:

$$\mathcal{X}\big(\mathsf{Gr\_Atoms}(\Gamma)\big) \;=\; \Big\{ X_\alpha \;\Big|\; \alpha \in \mathsf{Gr\_Atoms}(\Gamma) \Big\}.$$

We write $\mathcal{X}^{-1}$ for the inverse transformation, which is well-defined because $\mathcal{X}$ is one-one:

$$\mathcal{X}^{-1}(\Delta) \triangleq \Big\{ \pi[X_{\alpha_1} := \alpha_1, \ldots, X_{\alpha_n} := \alpha_n] \Big| \pi \in \Delta \text{ and } \{X_{\alpha_1}, \ldots, X_{\alpha_n}\} \text{ are the variables in } \pi \Big\},$$

where Δ is a set of propositional WFF's over the set of propositional variables $\mathcal{X}\big(\mathsf{Gr\_Atoms}(\Gamma)\big)$. If Γ is the singleton $\{\varphi\}$, we write $\mathcal{X}(\varphi)$ instead of $\mathcal{X}(\{\varphi\})$.

Lemma 30. *Let φ be an arbitrary first-order sentence and $\Gamma \triangleq \mathsf{Gr\_Expansion}\big(\boldsymbol{\Theta}_{\mathrm{pr,sk}}(\varphi)\big)$. Then φ is satisfiable (in the sense of first-order logic) iff $\mathcal{X}(\Gamma)$ is satisfiable (in the sense of propositional logic).*

Proof. By Lemma 28, φ is satisfiable iff Γ is satisfiable. It suffices therefore to show that: Γ is satisfiable (in the sense of first-order logic) iff $\mathcal{X}(\Gamma)$ is satisfiable (in the sense of propositional logic). Keep in mind that Γ is a set of quantifier-free sentences.

Let $\{\alpha_1, \alpha_2, \ldots\} = \mathsf{Gr\_Atoms}(\Gamma)$ the countable set, finite or infinite, of ground atoms occuring in Γ, and $\{X_{\alpha_1}, X_{\alpha_2}, \ldots\}$ the corresponding set of propositional variables occurring in $\mathcal{X}(\Gamma)$. For the left-to-right implication, assume there is a first-order structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$, and derive a truth-value assignment σ from $\mathcal{M}$ such that $\sigma \models \mathcal{X}(\Gamma)$. For the right-to-left implication, assume there is a truth-value assignment σ such that $\sigma \models \mathcal{X}(\Gamma)$, and derive a first-order structure $\mathcal{M}$ from σ such that $\mathcal{M} \models \Gamma$. All formal details omitted. $\square$

Exercise 31. Supply the missing formal details in the proof of Lemma 30. $\square$

If Γ is a set of first-order sentences, we write $\boldsymbol{\Theta}_{\text{pr,sk}}(\Gamma)$ to denote the set $\{\, \boldsymbol{\Theta}_{\text{pr,sk}}(\varphi) \mid \varphi \in \Gamma \,\}$. Similarly, Γ is a set of first-order sentences in Skolem form, we write $\mathsf{Gr\_Expansion}(\Gamma)$ to denote the set $\bigcup\{\, \mathsf{Gr\_Expansion}(\varphi) \mid \varphi \in \Gamma \,\}$.

Theorem 32 (Transfer Principle)**.** *Let Γ be a set, finite or infinite, of first-order sentences and let $\Gamma' = \mathsf{Gr\_Expansion}\big(\boldsymbol{\Theta}_{\text{pr,sk}}(\Gamma)\big)$ be the corresponding set of quantifier-free sentences. The following assertions hold:*

1. *Γ is satisfiable iff Γ' is satisfiable (both satisfiable in first-order logic).*
2. *If Γ is finitely satisfiable, then Γ' is finitely satisfiable (both satisfiable in first-order logic).*
3. *Γ' is satisfiable (in first-order logic) iff $\mathcal{X}(\Gamma')$ is satisfiable (in propositional logic).*
4. *Γ' is finitely satisfiable (in first-order logic) iff $\mathcal{X}(\Gamma')$ is finitely satisfiable (in propositional logic).*

Note that the converse implication in part 2 does not hold in general (why?).

Proof. We only sketch the proof, as most of the work is already done for the case when Γ is a singleton set. Parts 1 and 2 generalize Lemma 28, which depends on Lemmas 21 and 24. So, the proofs of these three previous lemmas have to be repeated with a set Γ of sentences, instead of a single sentence φ.

Parts 3 and 4 generalize Lemma 30, the proof of which depends on Lemma 28 only. So, for parts 3 and 4, we only need to generalize the proof of Lemma 30 for a set Γ of first-order sentences. $\square$

5 Compactness and Completeness for First-Order Logic

We first prove Compactness for first-order logic by invoking results of Herbrand theory. Then, in steps almost identical to the steps in Section 3, we prove Completeness as a consequence of Compactness.

Theorem 33 (Compactness for First-Order Logic)**.** *Let Γ be a set of first-order sentences. Then Γ is satisfiable iff Γ is finitely satisfiable.*

Proof. The left-to-right implication is immediate. For the converse, let Γ be finitely satisfiable. We use the *transfer principle* expressed by Theorem 32 and its notation.

If Γ is finitely satisfiable, then Γ' is finitely satisfiable (in first-order logic), by part 2 in the *transfer principle.* If Γ' is finitely satisfiable (in first-order logic), then $\mathcal{X}(\Gamma')$ is finitely satisfiable (in propositional logic), by part 4 in the *transfer principle.* If $\mathcal{X}(\Gamma')$ is finitely satisfiable (in propositional logic), then $\mathcal{X}(\Gamma')$ is satisfiable (in propositional logic) by Theorem 2, which is Compactness for propositional logic. If $\mathcal{X}(\Gamma')$ is satisfiable (in propositional logic), then Γ' is satisfiable (in first-order logic), by part 3 in the *transfer principle.* If Γ' is satisfiable (in first-order logic), then Γ is satisfiable (in first-order logic) by part 1 in the *transfer principle.* $\square$

Lemma 34. *Let* Γ *be a set of first-order sentences and* φ *an arbitrary first-order sentence. Then* $\Gamma \models \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is unsatisfiable – or, equivalently,* $\Gamma \not\models \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is satisfiable.*

Proof. Identical to the proof of Lemmas 6 and 16, except that here Γ is a set of first-order sentences and φ is a first-order sentence. □

Corollary 35. *Let* Γ *be a set of first-order sentences and* φ *an arbitrary first-order sentence. Then* $\Gamma \models \varphi$ *iff then there is a finite subset* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \models \varphi$*.*

Proof. Identical to the proof of Corollaries 7 and 17, except that here Γ is a set of first-order sentences and φ is a first-order sentence. Moreover, here we invoke Lemma 34 instead of Lemma 16, and Theorem 33 instead of Theorem 15. □

The next lemma is a weaker form of the Completeness Theorem for first-order logic. The Completeness Theorem for first-order logic in full generality is Theorem 38.

Lemma 36. *Let* $\varphi_1, \ldots, \varphi_n, \psi$ *be first-order sentences. If* $\varphi_1, \ldots, \varphi_n \models \psi$ *then* $\varphi_1, \ldots, \varphi_n \vdash \psi$*.*

Proof. The book [LCS] omits this lemma and its proof, though it mentions in passing that the natural-deduction proof system is "sound and complete" with respect to the formal semantics it discusses in Section 2.4 in details.[11] The proof can be carried out along the lines of the proof of Lemma 9, although the semantics of first-order logic are far more involved than the semantics of propositional logic. □

Exercise 37. Write the proof of Lemma 36 in detail. You will find it helpful to read the proof of the counterpart of this lemma for propositional logic in Section 1.4.4, pages 49-53, of the book [LCS]. □

Theorem 38 (Completeness for First-Order Logic)**.** *Let* Γ *be a set (possibly infinite) of first-order sentences, and* ψ *a first-order sentence. If* $\Gamma \models \psi$*, then* $\Gamma \vdash \psi$*.*

Proof. Identical to the proof of Theorems 10 and 20, except that all formulas are now first-order sentences. Moreover, we need to invoke Corollary 35 instead of Corollary 17, and Lemma 36 instead of Lemma 18. □

[11] See page 96 in Michael Huth and Mark Ryan, *Logic in Computer Science*, Second Edition, Cambridge University Press, 2004.