Formal Modeling with QBF's

Assaf Kfoury

February 18, 2017

1 A Two-Player Game: Chess

QBF's are convenient in modeling two-player games. Consider, for example, the problem of determining whether there is a winning strategy for a chess player in K steps:

(†) Assuming that White makes the first move, can White take the Black king in K steps regardless of Black's moves?

This problem can be modeled as a QBF. We ask if there exists a 1-st move of White, such that for all possible moves of Black that follow, there exists a 2-nd move of White, such that for all possible moves of Black that follow, if there exists a 3-rd move of White, such that ... and so forth, K times, such that the Black king is checkmated. The number of steps K has to be an odd natural, as White plays both the first and last move.

To formulate the problem, we first agree on some notations and conventions. There is a board of size $8 \times 8 = 64$. We denote the locations on the board by the integers: $\{1, 2, \ldots, 64\}$, as shown in Figure 1. To these 64 locations, we add one more, denoted by number 0, which is the location of a piece off the board. There is a total of 32 pieces, denoted by the integers: $\{1, 2, \ldots, 32\}$. We list some of the conventions and formulas we need to define:

- Propositional variable $x_{m,n,s}$ is set to *true* (resp. *false*) if piece m is (resp. is not) in location n at step s, where

  $$1 \leqslant m \leqslant 32, \quad 0 \leqslant n \leqslant 64, \quad \text{and} \quad 0 \leqslant s \leqslant K.$$

- I_0 is a quantifier-free WFF of propositional logic over the variables:

  $$\{\, x_{m,n,0} \mid 1 \leqslant m \leqslant 32,\ 0 \leqslant n \leqslant 64 \,\}$$

  that represent the **initial state** of the board.

- T^W_s is a quantifier-free WFF of propositional logic over the variables:

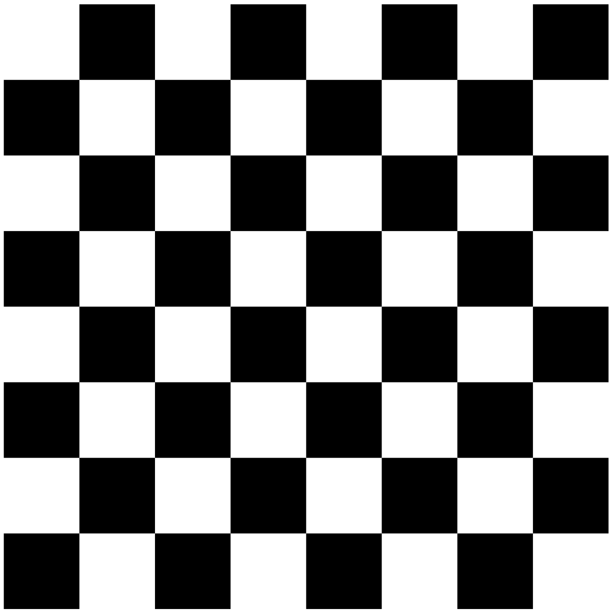
  $$\{\, x_{m,n,s} \mid 1 \leqslant m \leqslant 32,\ 0 \leqslant n \leqslant 64 \,\} \cup \{\, x_{m,n,s+1} \mid 1 \leqslant m \leqslant 32,\ 0 \leqslant n \leqslant 64 \,\}$$

  that represent the valid **moves** (or **transitions**) by White at step s, where $s = 0, 2, 4, \ldots, K-1$.

- T^B_s is a quantifier-free WFF of propositional logic over the variables:

  $$\{\, x_{m,n,s} \mid 1 \leqslant m \leqslant 32,\ 0 \leqslant n \leqslant 64 \,\} \cup \{\, x_{m,n,s+1} \mid 1 \leqslant m \leqslant 32,\ 0 \leqslant n \leqslant 64 \,\}$$

  that represent the valid **moves** (or **transitions**) by Black at step s, where $s = 1, 3, 5, \ldots, K-2$.



57	58	59	60	61	62	63	64
49	50	51	52	53	54	55	56
41	42	43	44	45	46	47	48
33	34	35	36	37	38	39	40
25	26	27	28	29	30	31	32
17	18	19	20	21	22	23	24
9	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8

Figure 1: A chess board (on the left) and the numbering of its squares (on the right). Initially, all White pieces are placed in squares $1, 2, \ldots, 16$ and all Black pieces are placed in squares $49, 50, \ldots, 64$.

- G_K is a quantifier-free WFF of propositional logic over the variables:

  $$\{\, x_{m,n,K} \mid 1 \leqslant m \leqslant 32,\ 0 \leqslant n \leqslant 64 \,\}$$

  representing the **goal**, *i.e.*, at step K, White king (resp. Black king) is on (resp. off) the board.

Are the preceding conventions sufficient to encode question (†) in the form of a QBF? Let's try.

We first try to write I_0, which is supposed to represent the initial state of the board. It becomes immediately evident that we need to agree on further conventions:

- Let $\{1, 2, \ldots, 16\}$ be the numbers of the White pieces, initially placed in locations $\{1, 2, \ldots, 16\}$.
- Let $\{17, 18, \ldots, 32\}$ be the numbers of the Black pieces, initially placed in locations $\{49, 50, \ldots, 64\}$.

With the preceding convention, locations $\{17, 18, \ldots, 48\}$ are in the middle part of the board and not occupied by any piece initially, White or Black.

At this point we do not need to distinguish between the different kinds of pieces (king, queen, rook, knight, bishop, pawn). The game is in its initial state iff I_0 evaluates to *true*, where:

$$\begin{array}{llll}
I_0 \triangleq & \bigwedge \{\, \neg\, x_{m,0,0} \mid 1 \leqslant m \leqslant 32 \,\} & \wedge & \text{(all the pieces are initially on the board)} \\
& \bigwedge \{\, x_{m,n,0} \mid 1 \leqslant m \leqslant 16 \ \&\ n = m \,\} & \wedge & \text{(initial positions of White pieces)} \\
& \bigwedge \{\, \neg\, x_{m,n,0} \mid 1 \leqslant m \leqslant 16 \ \&\ n \neq m \,\} & \wedge & \text{(no White pieces elsewhere)} \\
& \bigwedge \{\, x_{m,n,0} \mid 17 \leqslant m \leqslant 32 \ \&\ n = m + 32 \,\} & \wedge & \text{(initial positions of Black pieces)} \\
& \bigwedge \{\, \neg\, x_{m,n,0} \mid 17 \leqslant m \leqslant 32 \ \&\ n \neq m + 32 \,\} & & \text{(no Black pieces elsewhere)}
\end{array}$$

Make sure you understand, and agree with, the formulation of I_0. The variables mentioned in I_0 are all the variables whose third index is 0, *i.e.*, all the variables $x_{m,n,0}$ for some m and n.

For succintness, we introduce some shorthands. If we write $\{x_{m,n,s}\}$, this is a shorthand representation of the set of all variables, *i.e.*:

$$\{x_{m,n,s}\} \triangleq \{\, x_{m,n,s} \mid 1 \leqslant m \leqslant 32,\ 0 \leqslant n \leqslant 64,\ 0 \leqslant s \leqslant K \,\}$$

Similarly, if we write $\{x_{m,n,0}\}$, this is a shorthand representation of all the variables at step 0, *i.e.*:

$$\{x_{m,n,0}\} \triangleq \{\, x_{m,n,0} \mid 1 \leqslant m \leqslant 32,\ 0 \leqslant n \leqslant 64 \,\}$$

If Y is a set of propositional variables $\{y_1, y_2, \ldots, y_p\}$, then "$\exists\, Y$" is a shorthand defined by:

$$\exists\, Y \triangleq \exists y_1 \exists y_2 \cdots \exists y_p$$

There is exactly one valuation $\mathcal{I} : \{x_{m,n,0}\} \to \{\mathit{true}, \mathit{false}\}$ which makes I_0 *true*, corresponding to the initial state of the chess game. We can express this as a closed QBF:

$$\exists\, \{x_{m,n,0}\}.\, I_0$$

whose validity asserts the existence of a state that satisfies precisely the conditions of the initial state of the chess game.[1]

Exercise 1 Write a (quantifier-free) propositional WFF, call it $\varphi_{m,n,s}$, over the set of variables $\{\, x_{m,n,s} \mid 0 \leqslant n \leqslant 64 \,\}$, which expresses the fact that a piece m is placed in *at most one* location n (in fact, in *exactly one* location n, with location 0 being "off the board") at step s. Specifically, $\varphi_{m,n,s}$ evaluates to *true* (under any valuation $\mathcal{I}$) iff at step s:

- piece m is placed in location n, and
- piece m is **not** placed in location n' for every $n' \in \{0, 1, \ldots, 64\} - \{n\}$. □

Exercise 2 Write a (quantifier-free) propositional WFF, call it ψ_s, which expresses the fact that no two pieces are found in the same board location at step s. (Note, however, two or more pieces can be found in the location "off the board" at the same step i.) Specifically, ψ_s evaluates to *true* (under any valuation $\mathcal{I}$) iff at step s:

- for every piece m and every board location n,
  if m is placed in n, then every piece $m' \neq m$ is not placed in n.

Hint: You may find it useful to use $\varphi_{m,n,s}$ from Exercise 1. □

Exercise 3 Define the (quantifier-free) propositional WFF G_K over the set of variables $\{\, x_{m,n,K} \,\}$ expressing the fact that White king is on the board and Black king is off the board at step K.

Hint 1: You may find it useful to use $\varphi_{m,n,s}$ from Exercise 1.
Hint 2: You need to specify numbers for the White king and Black king. Take 5 for the White king (initially placed in location 5) and 29 for the Black king (initially placed in location 61). □

Exercise 4 Define the (quantifier-free) propositional WFF T_s^W (resp. T_s^B) that expresses a legal move for White (resp. for Black) at step s. Keep in mind the following:

[1] As remarked in lecture, *validity* and *satisfiability* of *closed* QBF's coincide, which is not the case for *open* QBF's. Also, *validity* of a *closed existential* QBF in prenex form is equivalent to *satisfiability* of its (quantifier-free) matrix.

- T_s^W (resp. T_s^B) must reflect that there is exactly one legal move at step s, *i.e.*, there cannot be two distinct legal moves in two different parts of the board simultaneously.
- A legal move by White (resp. Black) may lead to Black's (resp. White's) loss of one of its pieces, *i.e.*, the lost piece is "moved to location 0". The formula T_s^W (resp. T_s^B) should reflect this fact.

Hint 1: You may find it useful to use the propositional WFF's defined in Exercises 1 and 2.
Hint 2: You need to be more specific in assigning numbers to pieces. Take 1 for the number of a White rook, 2 for the number of a White knight, 3 for the number of a White bishop, ..., and 32 for the number of a Black rook. □

Exercise 5 Define a closed QBF Φ_K in prenex form that encodes our question (†) in the opening paragraph. Φ_K should look like this:

$$\Phi_K \triangleq \exists\{x_{m,n,0}\}\,\exists\{x_{m,n,1}\}\,\forall\{x_{m,n,2}\}\,\exists\{x_{m,n,3}\}\,\forall\{x_{m,n,4}\}\,\cdots\,\forall\{x_{m,n,K-2}\}\,\exists\{x_{m,n,K-1}\}\,\exists\{x_{m,n,K}\}.\ \Psi_K$$

and should be *valid* if the answer to (†) is YES, and not *valid* if the answer to (†) is NO. The matrix Ψ_K of Φ_K is a quantifier-free WFF of propositional logic. The leading existential quantifier "$\exists\{x_{m,n,0}\}$" and the trailing existential quantifier "$\exists\{x_{m,n,K}\}$" are not really necessary; they are inserted in order to make Φ_K a closed formula, so that we can pass it on to a QBF solver (which typically requires its input formula to be closed).

Hint: You will find it useful to use the propositional WFF's defined in the preceding exercises. □

Exercise 6 This is an open-ended implementation exercise. Download a open-source QBF solver of your choice (some of them are available from the website `http://www.satlive.org/solvers/`) and run it to determine whether the closed QBF Φ_K you define in Exercise 5 is valid. For small values of K, your QBF solver will determine Φ_K is not valid, but for relatively large values of K, it will probably exhaust your laptop (or desktop). Try to restrict the legal moves of Black, by appropriately redefining T_s^B, so that your QBF solver will terminate in timely fashion and confirm that Φ_K is valid. (For a start, you may try to modify T_s^B so that it only represents legal moves of Black pawns, *i.e.*, Black is a dim-witted player who will not move any of the rooks, the knights, the bishops, the king, and the queen.) □

2 A Two-Player Game: Tic-Tac-Toe

This is a continuation of the examination in the previous handout, *Formal Modeling with Propositional Logic* (click here), of Tic-Tac-Toe on a $K \times K$ board, with $K \geqslant 3$.

In the previous examination, we could model the initial configuration and winning configurations using PL, as well as any particular configuration between the first and the last in the game. What we did not try, because it is difficult, is to formally represent the transition from one configuration to the next. To do this, we need to introduce a way of expressing the board configuration in any particular step and how it relates to the board configuration in the very next step. We do this by introducing a third index, call is "s" for "step", in addition to indices i and j which are used to identify locations on the board.

In anticipation of our use of quantifiers, we change the notation from propositional atoms $P_{i,j}$ and $Q_{i,j}$ to propositional variables $x_{i,j,s}$ and $y_{i,j,s}$, with the latter now including a third index s. Index $s = 0$ refers to the initial board configuration, corresponding to "step 0". And $s = K^2$ refers to the final board configuration, under the assumption that the game is pursued for exactly K^2 even if the X-player reaches a winning configuration before, with $[K]$ as a shorthand for the set $\{1, \ldots, K\}$:

- Use three-indexed propositional variables, $x_{i,j,s}$ and $y_{i,j,s}$ with $i, j \in [K]$ and $s \in \{0, 1, \ldots, K^2\}$ to identify the squares where X and O are located on the board at step s. Specifically,

$$x_{i,j,s} = \begin{cases} \mathit{true} & \text{if square } (i,j) \in [K] \times [K] \text{ contains X at step } s, \\ \mathit{false} & \text{if square } (i,j) \in [K] \times [K] \text{ does not contain X at step } s, \end{cases}$$

$$y_{i,j,s} = \begin{cases} \mathit{true} & \text{if square } (i,j) \in [K] \times [K] \text{ contains O at step } s, \\ \mathit{false} & \text{if square } (i,j) \in [K] \times [K] \text{ does not contain O at step } s. \end{cases}$$

We are therefore using $K^2 \times (1 + K^2)$ variables $x_{i,j,s}$, and another $K^2 \times (1 + K^2)$ variables $y_{i,j,s}$, for a total of $2K^2 + 2K^4$ variables. This is a relatively large number to manipulate (certainly for a human user) – hence, the importance of automating whatever questions we may want to ask about the game.

We assume that the game always starts with the X-player making the first move. Thus, the X-player makes a move at the even-numbered steps, and the Y-player makes a move at the odd-numbered steps. When K is odd, the X-player makes one more move than the Y-player to reach the end of the game; when K is even, the two players make an equal number of moves.

Exercise 7 This is a warm-up and easy exercise, based on the discussion of Tic-Tac-Toe in the previous handout, *Formal Modeling with Propositional Logic* (click here). You now have to keep track of the appropriate value for the third index s.

1. Write a quantifier-free QBF, *i.e.*, a propositional WFF, φ_{start} which formally models the starting configuration: every valuation of the variables that satisfies φ_{start} corresponds to the initial configuration (empty board).

2. Write a quantifier-free QBF, *i.e.*, a propositional WFF, $\varphi_{\mathsf{x\text{-}win}}$ which formally models a winning configuration for the X-player: every valuation of the variables that satisfies $\varphi_{\mathsf{x\text{-}win}}$ corresponds to a winning configuration for X-player.

3. Write a quantifier-free QBF, *i.e.*, a propositional WFF, $\varphi_{\mathsf{o\text{-}win}}$ which formally models a winning configuration for the O-player: every valuation of the variables that satisfies $\varphi_{\mathsf{o\text{-}win}}$ corresponds to a winning configuration for O-player.

   Hint: See Exercise 3 in the handout *Formal Modeling with Propositional Logic*. $\varphi_{\mathsf{o\text{-}win}}$ is not a "mirror image" of $\varphi_{\mathsf{x\text{-}win}}$ whereby the variables $x_{i,j,s}$ and $y_{i,j,s}$ are interchanged. □

We need to formally represent the legal moves of Tic-Tac-Toe, in going from step s to step $s + 1$, for every $s \in \{0, 1, \ldots, K^2 - 1\}$. To this end, we define:

- θ_s^{X} is a quantifier-free QBF over the variables:

  $$\{\, x_{i,j,s} \mid i, j \in [K] \,\} \cup \{\, y_{i,j,s} \mid i, j \in [K] \,\} \cup \{\, x_{i,j,s+1} \mid i, j \in [K] \,\} \cup \{\, y_{i,j,s+1} \mid i, j \in [K] \,\}$$

  that represent the legal **moves** by the X-player at step s, where
  $s = 0, 2, 4, \ldots, K^2 - 2$ (if K^2 is even), $K^2 - 1$ (if K^2 is odd).

- θ_s^{O} is a is quantifier-free QBF over the variables:

  $$\{\, x_{i,j,s} \mid i, j \in [K] \,\} \cup \{\, y_{i,j,s} \mid i, j \in [K] \,\} \cup \{\, x_{i,j,s+1} \mid i, j \in [K] \,\} \cup \{\, y_{i,j,s+1} \mid i, j \in [K] \,\}$$

  that represent the legal **moves** by the O-player at step s, where
  $s = 1, 3, 5, \ldots, K^2 - 1$ (if K^2 is even), $K^2 - 2$ (if K^2 is odd).

Exercise 8 Write the details of the quantifier-free QBF's θ_s^{X} and θ_s^{O}:

1. θ_s^{X} is satisfied by a valuation of the variables iff the valuation describes a legal move of the X-player at step s.
2. θ_s^{O} is satisfied by a valuation of the variables iff the valuation describes a legal move of the Y-player at step s.

Hint: If $\mathcal{I}$ is a valuation of the variables in $\{\, x_{i,j,s} \mid i,j \in [K]\,\} \cup \{\, y_{i,j,s} \mid i,j \in [K]\,\}$ representing a valid configuration at step s, write θ_s^{X} (resp. θ_s^{O}) so that there cannot be two distinct valuations $\mathcal{I}_1$ and $\mathcal{I}_2$ of the variables in $\{\, x_{i,j,s+1} \mid i,j \in [K]\,\} \cup \{\, y_{i,j,s+1} \mid i,j \in [K]\,\}$ such that both $\mathcal{I} \cup \mathcal{I}_1$ and $\mathcal{I} \cup \mathcal{I}_2$ satisfy θ_s^{X} (resp. θ_s^{O}). In other words, θ_s^{X} (resp. θ_s^{O}) must reflect the fact that, given that $(s-1)$ legal steps are already completed, the s-th step cannot consist of two distinct legal moves. $\square$

Answer for Exercise 8:

$$\theta_s^{\mathsf{X}} := \bigvee_{(i,j)\in[K]\times[K]} \Big(\bigwedge_{(k,\ell)\in[K]\times[K]-\{(i,j)\}} (x_{k,\ell,s} \leftrightarrow x_{k,\ell,s+1}) \;\wedge \bigwedge_{(k,\ell)\in[K]\times[K]-\{(i,j)\}} (y_{k,\ell,s} \leftrightarrow y_{k,\ell,s+1}) \;\wedge\; \neg x_{i,j,s} \;\wedge\; \neg y_{i,j,s} \;\wedge\; x_{i,j,s+1} \;\wedge\; \neg y_{i,j,s+1} \Big)$$

$$\theta_s^{\mathsf{O}} := \bigvee_{(i,j)\in[K]\times[K]} \Big(\bigwedge_{(k,\ell)\in[K]\times[K]-\{(i,j)\}} (x_{k,\ell,s} \leftrightarrow x_{k,\ell,s+1}) \;\wedge \bigwedge_{(k,\ell)\in[K]\times[K]-\{(i,j)\}} (y_{k,\ell,s} \leftrightarrow y_{k,\ell,s+1}) \;\wedge\; \neg x_{i,j,s} \;\wedge\; \neg y_{i,j,s} \;\wedge\; \neg x_{i,j,s+1} \;\wedge\; y_{i,j,s+1} \Big)$$

Make sure you understand the formulation of θ_s^{X} and θ_s^{O} before proceeding further in this handout.

For the next exercise, we define winning strategies in Tic-Tac-Toe. The X-player is always the one who makes the first move. For the case "K^2 is even" we define (the case "K^2 is odd" is left to you):

- A *winning strategy for the X-player* means: there is an X-move, for all O-moves, there is an X-move, for all O-moves, …, for all O-moves, it is the case that the X-player wins.
- A *winning strategy for the O-player* means: for all X-moves, there is an O-move, for all X-moves, there is an O-move, …, there is an O-move, such that the X-player does not win.

For succintness, we introduce some shorthands. If we write $\{x_{i,j,0}\}$, this is a shorthand representation of the set of all variables at step 0, *i.e.*:

$$\{x_{i,j,0}\} \triangleq \{\, x_{i,j,0} \mid i,j \in [K]\,\}$$

Similarly, if we write $\{x_{i,j,1}\}$, $\{x_{i,j,2}\}$, etc., and $\{y_{i,j,0}\}$, $\{y_{i,j,1}\}$, $\{y_{i,j,2}\}$, etc.

Exercise 9 Define a closed QBF $\Phi_{\textsf{X-strat}}$ in prenex form which is valid iff X-player has a winning strategy. $\Phi_{\textsf{X-strat}}$ should look like this, when K^2 is even:

$$\begin{aligned}
\Phi_{\textsf{X-strat}} \triangleq\ & \exists\ \{x_{i,j,0}\} \cup \{y_{i,j,0}\}\ .\\
& \exists\ \{x_{i,j,1}\} \cup \{y_{i,j,1}\}\ .\\
& \forall\ \{x_{i,j,2}\} \cup \{y_{i,j,2}\}\ .\\
& \exists\ \{x_{i,j,3}\} \cup \{y_{i,j,3}\}\ .\\
& \forall\ \{x_{i,j,4}\} \cup \{y_{i,j,4}\}\ .\\
& \ \cdots\\
& \forall\ \{x_{i,j,K^2-2}\} \cup \{y_{i,j,K^2-2}\}\ .\\
& \exists\ \{x_{i,j,K^2-1}\} \cup \{y_{i,j,K^2-1}\}\ .\\
& \forall\ \{x_{i,j,K^2}\} \cup \{y_{i,j,K^2}\}\ .\ \ \Psi_{\textsf{X-strat}}
\end{aligned}$$

The leading existential quantification "$\exists\{x_{i,j,0}\}\cup\{y_{i,j,0}\}$" is not really necessary; it is inserted in order to make $\Phi_{\textsf{X-strat}}$ a closed formula, ready to be passed on to a QBF solver (that requires its input formula to be closed). When K^2 is odd, the last three quantifications in $\Phi_{\textsf{X-strat}}$ should be:

$$\begin{aligned}
\Phi_{\textsf{X-strat}} \triangleq\ & \\
& \ \cdots\\
& \exists\ \{x_{i,j,K^2-2}\} \cup \{y_{i,j,K^2-2}\}\ .\\
& \forall\ \{x_{i,j,K^2-1}\} \cup \{y_{i,j,K^2-1}\}\ .\\
& \exists\ \{x_{i,j,K^2}\} \cup \{y_{i,j,K^2}\}\ .\ \ \Psi_{\textsf{X-strat}}
\end{aligned}$$

When K^2 is odd, the leading and last quantifications "$\exists\{x_{i,j,0}\}\cup\{y_{i,j,0}\}$" and "$\exists\{x_{i,j,K^2}\}\cup\{y_{i,j,K^2}\}$" are not really necessary and are inserted in order to make $\Phi_{\textsf{X-strat}}$ a closed formula.

Whether K^2 is even or odd, write the matrix $\Psi_{\textsf{X-strat}}$ as a quantifier-free formula in terms of the two formulas $\varphi_{\textsf{start}}$ and $\varphi_{\textsf{x-win}}$ in Exercise 7, and the K^2 formulas $\theta_s^{\textsf{X}}$ and $\theta_s^{\textsf{O}}$ in Exercise 8. □

<u>Answer for Exercise 9</u>: Consider the case when K^2 is even. We give a preliminary, hopefully easier to understand, answer $\Phi'_{\textsf{X-strat}}$ before we turn it into the final desired answer $\Phi_{\textsf{X-strat}}$.

$$\begin{aligned}
&\Phi'_{\textsf{X-strat}} \triangleq\\
&\exists\{x_{i,j,0}\}\cup\{y_{i,j,0}\}.\ \varphi_{\textsf{start}}\ \wedge\ \exists\ \{x_{i,j,1}\} \cup \{y_{i,j,1}\}.\ \ \theta_0^{\textsf{X}}\ \wedge\\
&\qquad\qquad\qquad\qquad\qquad\quad \forall\ \{x_{i,j,2}\} \cup \{y_{i,j,2}\}.\ \ \theta_1^{\textsf{O}}\ \rightarrow\\
&\qquad\qquad\qquad\qquad\qquad\quad \exists\ \{x_{i,j,3}\} \cup \{y_{i,j,3}\}.\ \ \theta_2^{\textsf{X}}\ \wedge\\
&\qquad\qquad\qquad\qquad\qquad\quad \forall\ \{x_{i,j,4}\} \cup \{y_{i,j,4}\}.\ \ \theta_3^{\textsf{O}}\ \rightarrow\\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdots\\
&\qquad\qquad\qquad\qquad\qquad\quad \exists\ \{x_{i,j,K^2-1}\} \cup \{y_{i,j,K^2-1}\}.\ \ \theta_{K^2-2}^{\textsf{X}}\ \wedge\\
&\qquad\qquad\qquad\qquad\qquad\quad \forall\ \{x_{i,j,K^2}\} \cup \{y_{i,j,K^2}\}.\ \ \theta_{K^2-1}^{\textsf{O}}\ \rightarrow\ \varphi_{\textsf{x-win}}
\end{aligned}$$

Note that, at step $K^2 - 1$, it is the O-player's turn, and there is only one square on the board without an X and without an O. All that the O-player can now do is to place an O on the sole empty square. This should be reflected by the formula $\theta^{\mathsf{O}}_{K^2-1}$. Also, note that $\Phi'_{\mathsf{X\text{-}strat}}$ is closed but not in prenex form. We can transform $\Phi'_{\mathsf{X\text{-}strat}}$ into prenex form, with the resulting QBF being the final desired $\Phi_{\mathsf{X\text{-}strat}}$ whose (quantifier-free) matrix is:

$$\Psi_{\mathsf{X\text{-}strat}} \triangleq \varphi_{\mathsf{start}} \wedge \theta^{\mathsf{X}}_0 \wedge \Big(\theta^{\mathsf{O}}_1 \rightarrow \Big(\theta^{\mathsf{X}}_2 \wedge \big(\theta^{\mathsf{O}}_3 \rightarrow \cdots \big(\theta^{\mathsf{X}}_{K^2-2} \wedge \big(\theta^{\mathsf{O}}_{K^2-1} \rightarrow \varphi_{\mathsf{x\text{-}win}}\big)\big)\big)\Big)\Big)$$

Convince yourself $\Psi_{\mathsf{X\text{-}strat}}$ is correctly defined, based on the transformation from $\Phi'_{\mathsf{X\text{-}strat}}$ to $\Phi_{\mathsf{X\text{-}strat}}$.

Exercise 10 Define a closed QBF $\Phi_{\mathsf{O\text{-}strat}}$ in prenex form which is valid iff O-player has a winning strategy.

Hint: Do Exercise 9 first. □