

THE ANGEL WINS

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ABSTRACT. The angel-devil game is played on an infinite two-dimensional “chessboard” \mathbb{Z}^2 . The squares of the board are all white at the beginning. The players called angel and devil take turns in their steps. When it is the devil’s turn, he can turn a square black. The angel always stays on a white square, and when it is her turn she can fly at a distance of at most J steps (each of which can be horizontal or vertical) steps to a new white square. Here J is a constant. The devil wins if the angel does not find any more white squares to land on. The result of the paper is that if J is sufficiently large then the angel has a strategy such that the devil will never capture her. This deceptively easy-sounding result has been a conjecture, surprisingly, for about thirty years. Several other independent solutions have appeared also in this summer: see the Wikipedia. Some of them prove the result for an angel that can make up to two steps (including diagonal ones).

The solution opens the possibility to solve a number of related problems and to introduce new, adversarial concepts of connectivity.

1. INTRODUCTION

The angel-devil game is played on an infinite two-dimensional “chessboard” \mathbb{Z}^2 . The squares of the board are all white at the beginning. The players called angel and devil take turns in their steps. When it is the devil’s turn, he can turn a square black. The angel always stays on a white square, and when it is her turn she can fly at a distance of at most J (horizontal or vertical) steps to a new white square. Here J is a constant. The devil wins if the angel does not find any more white squares to land on. The result of the paper is that if J is sufficiently large then the angel has a strategy such that the devil will never capture her. This solves a problem that to the authors’ knowledge has been open for about 30 years.

See the bibliography and also the Wikipedia item on the “angel problem”.

1.1. Weights. Let us make the devil a little stronger. Instead of jumping a distance J in one step, assume that the angel makes at most one (vertical or horizontal) step at a time, but the devil can deposit only a weight of size $\sigma = 1/J$ in one step. The angel is not allowed to step on a square with weight ≥ 1 . We do not restrict the devil in how he distributes this weight, it need not be in fractions of size $1/J$.

Definition 1.1. Let $\mu(S) = \mu_t(S)$ be the weight (measure) of set S at time t . The devil’s restriction is

$$\mu_{t+1}(\mathbb{Z}^2) \leq \mu_t(\mathbb{Z}^2) + \sigma.$$

Let \mathcal{M} be the set of all measures. ┘

The main theorem of the paper is the following.

Theorem 1. *For sufficiently small σ , the angel has a strategy in which she will never run out of places to land on.*

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1.2. Informal idea.

Definition 1.2. Let

$$Q > 1$$

be an integer parameter. For an integer $k \geq 0$ we call a square a k -square, or k -colony if the coordinates of its corners are multiples of Q^k . A $(k + 1)$ -square can be broken up into rows each of which is the union of Q disjoint k -squares. It can also be broken up into columns. The side length of a square U is denoted $|U|$. We will call a square U *bad* (for the current measure μ), if

$$\mu(U) \geq |U|.$$

Otherwise it is called *good*. ┘

The angel needs a strategy that works on all scales, to make sure she is not surrounded in the short term as well as in the long term. Let us try to develop a strategy for her on the scale of $(k + 1)$ -colonies, while taking for granted certain possibilities for her on the scale of k -colonies. In the context of $(k + 1)$ -colonies the k -colonies will be called *cells*.

A $(k + 1)$ -colony S^* can be broken up into rows R_1, \dots, R_Q . If S^* is good then we have $\mu(R_i) < Q^k$ for at least one $i \in \{1, \dots, Q\}$. In this row there is not even a single bad cell, and at most one cell can be close to badness, the other ones will be “safe”. If the row is “very” good then even the single square close to badness will not be spoiled soon.

Suppose that another $(k + 1)$ -colony D^* is adjacent to S^* on the north, and that

- (a) Even the two $(k + 1)$ -colonies jointly are “very good”: $\mu(S^* \cup D^*) < (1 - \delta)Q^{k+1}$. Then there is a “good” column $C(1)$.
- (b) S^* is far from badness: we will call it “clean”. It has a clean row $R'(0)$ the angel is in it.
- (c) D^* is even farther from badness: we will call it “safe”.

Then we may be able to pass from S^* to D^* along column $C(1)$, passing to it in row $R'(0)$. By the time we arrive into D^* it may not be safe anymore but it is still clean, and we can pass into a clean row $R''(1)$ in it. This simple scheme of passing from one big colony to the next will be called a *step*.

This scheme has many holes yet, and we will fix them one-by-one, adding new and new complications. Fortunately the process converges. Let us look at some of the issues.

1. The digression along row $R'(0)$ to get into column $C(1)$ causes a delay. If our delays are not under control (especially in a recursive scheme like ours) then the devil gains too much time and can put down too much weight. Let U be a path of cells (viewed as a union of cells). We will bound the time along it by the following expression:

$$\tau_{\text{gc}}(U) + \rho\mu(U).$$

The term $\tau_{\text{gc}}(U)$ is essentially the sum of the lengths of the straight runs of cells in U : we call it the *geometric cost*. Now when the angel makes a digression along row $R'(0)$ to get to column $C(1)$ it had a reason *outside* its path of cells: namely, she could not pass straight in column $C(0)$, because of some weight $\mu(C(0) \setminus R'(0))$ in this column. We will upperbound the extra geometric cost by $\rho\mu(C(0) \setminus R'(0))$. So the above sum can be estimated as

$$\tau_{\text{gc}}(U^*) + \rho\mu(C(0) \setminus R'(0)) + \rho\mu(U) \leq \tau_{\text{gc}}(U^*) + \rho\mu(U^*).$$

This is how we “make the devil pay” for causing a digression: the time bound formula is conserved when passing from the lower to the higher scale.

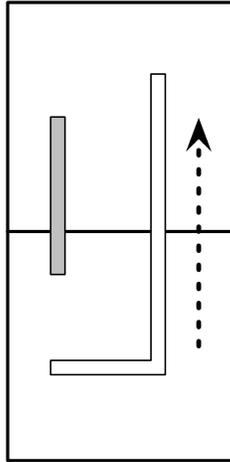


FIGURE 1. The white path is the path taken by the angel to get from the lower big square to the upper one. The grey area is to blame that the angel did not go straight.

2. Even in a clean row there is possibly one cell that is not safe, in which we do not want to “land” but which we need to “pass through”. So we have to look at the situation of three $(k + 1)$ -colonies on top of each other, the bottom one, S^* still clean, the top one, D^* still safe, but about the middle one we know only that the union of the three big colonies together is good. It is in the nature of the game that we will have to tackle a situation like this without advance guarantee that we will be able to arrive at the top. We have to attempt it and may fail, ending up possibly where we started (but then knowing that the devil had to spend a lot of his capital on this). The strategy will be essentially to try to pass in each column. Because of the cleanness of S^* there is always a clean row to fall back to. This scheme of attempted passage will be called an *attack*. Its implementation uses attacks on the cell level.
3. In the implementation of attacks, we must be careful about the idea of “falling back to” a clean row, since the time bound introduced above assumes that the path we use is *not self-intersecting*. The attack must be implemented with some care to achieve this.¹ We introduce some primitive “moves” for the angel that incorporate some of the possible retreating steps. Also, to avoid retreat when it is not really needed, we introduce the notion of a “continuing attack”, which continues from the result of a failed attack in the eastern neighbor column without falling back.
4. With the extra kinds of move a self-nonintersecting implementation can be achieved, but there is still a problem. The delay (extra geometric cost) of the attacks cannot always be charged directly to some mass outside the path of cells. Fortunately the case when it cannot is an attack on the level of $(k + 1)$ -colonies containing many failed attacks on the level of k -colonies. A failed attack has a lot of mass *inside it*, and part of it can be used to pay for the extra geometric cost. In our new time bound formula therefore if there are n

¹ It is a result in [1] that if the angel has a winning strategy she also has a self-nonintersecting one. It is interesting that the strategy we develop is self-nonintersecting for an apparently different reason.

failed attacks they will contribute a “profit” (negative cost):

$$\tau(U) = \tau_{\text{gc}}(U) + \rho_1 \mu(U) - \rho_2 n Q^k.$$

This will be sufficient to account for all the digressions.

2. CONCEPTS

Let us proceed to the formal constructions.

2.1. AD-games. For our hierarchical solution, we generalize the angel-devil game into a game called AD-game.

2.1.1. Parameters.

Definition 2.1. A union $U = U_1 \cup \dots \cup U_n$ of (horizontally or vertically) adjacent squares of equal size will be called a *run*. We write

$$|U| = n|U_1|.$$

┘

Definition 2.2. Let

$$2/3 < \xi < 1, \quad 0 < \delta < \xi/2, \quad 0 < \sigma, \quad \rho_1 > \rho_2 > 0 \tag{1}$$

be real parameters to be fixed later. We say that the run $U = U_1 \cup \dots \cup U_n$ is *bad* for measure μ if $\mu(U) \geq |U_1|$, otherwise it is good. We will say that it is *i-good* if

$$\mu(U) < (1 - i\delta)|U_1|.$$

In particular, 0-good means simply *good*. Similarly, we will call a run *i-safe* if $\mu(U) < (\xi - i\delta)|U_1|$. ┘

Definition 2.3. For an integer $B > 0$ and an integer x we write

$$\lfloor x \rfloor_B = B \cdot \lfloor x/B \rfloor.$$

Similarly for a vector $\mathbf{u} = (x, y)$ with integer coordinates,

$$\lfloor \mathbf{u} \rfloor_B = B \cdot (\lfloor x/B \rfloor, \lfloor y/B \rfloor).$$

The set

$$P = \{(0, 1), (0, -1), (1, 0), (-1, 0)\}.$$

will be called the set of *directions*. These directions will also be called *east*, *west*, *north*, *south*. ┘

2.1.2. The structure.

Definition 2.4. An AD-game \mathbb{G} with colony size B consists of steps alternating between a player called angel and another one called devil. At any one time, the current configuration determines the possibilities open for the player whose turn it is. We will only consider strategies of the angel. More precisely, a game

$$\mathbb{G} = \mathbb{G}(B, \Delta_a, \Delta_d) \tag{2}$$

is defined as follows. As before, the devil controls a measure μ_t and can add the amount $\sigma > 0$ to the total mass at each time, so that $\mu_{t+1} \geq \mu_t$ and $\mu_{t+1}(\mathbb{Z}^2) - \mu_t(\mathbb{Z}^2) \leq \sigma$.

The plane is partitioned into a lattice of squares of size B called *colonies*. Point \mathbf{u} is contained in the colony

$$\mathcal{B}(\mathbf{u}) = \lfloor \mathbf{u} \rfloor_B + \{0, \dots, B-1\}^2.$$

┘

The game is played using the following definitions.

Definition 2.5. The game follows a sequence of moves $r = 1, 2, \dots$, associated with an increasing sequence of integer times t_r . At move r , times t_1, \dots, t_r are already defined, the angel stays at a point $\mathbf{p}_r \in \mathcal{B}(\mathbf{p}_r)$. The triple

$$(t_r, \mathbf{p}_r, \mu_{t_r})$$

will be called an *essential configuration* of the game in step r . Let us call *default essential configuration* the configuration $(0, (0, 0), 0)$, that is the configuration at time 0, position at the origin $(0, 0)$, the null measure.

The game starts at time $t_0 = 0$ from position $\mathbf{p}_0 = (0, 0)$ with initial measure $\mu_0 = 0$. The position

$$\mathbf{w}_r = \lfloor \mathbf{p}_r / B \rfloor$$

will be called the *colony position* at step r . ┘

2.1.3. *Moves and the angel's constraints.* Let us see what are the potential moves.

Definition 2.6. There is a finite set

$$\Pi$$

of symbols called *potential moves*. Each move z has finite sets

$$\mathcal{E}(z) \subset \mathcal{H}(z) \subset \mathbb{Z}^2$$

where $\mathcal{H}(z)$ is called the *template* of the move, and \mathcal{E} is called the set of *end positions* in the template. There is also an element $\text{dest}(z) \in \mathcal{E}(z)$ called the *destination position* of z . ┘

Let us see how moves will be used before defining them.

Definition 2.7. At any time t_r , when staying in some *start colony* S , the angel chooses a *move* $z = z_r$ from the set

$$\Delta_a(\mathbf{p}_r, \mu_{t_r}) \subset \Pi.$$

She loses if this set is empty. Let us call t_r the *start time* of the move and t_{r+1} the *end time* (unknown yet) of the move. The *body* of the move is the set

$$M = M(\mathbf{w}, z) = S + B \cdot \mathcal{H}(z).$$

Colonies $\mathbf{w} + \mathbf{u}$ for $\mathbf{u} \in \mathcal{E}(z)$ are the possible *end colonies* of the move. There will be several more restrictions on the moves the angel can choose. The devil will deposit the angel at a point in a certain end colony, at the endtime of the move chosen by him. There will be several restrictions on the devil's choice of place and time. ┘

Definition 2.8. A pair

$$\mathbf{a} = (\mathbf{w}, z)$$

where $\mathbf{w} \in \mathbb{Z}^2$ and $z \in \Pi$ is called a *located move*. We call the colony $\mathcal{B}(\mathbf{w})$ where the angel stays at the beginning of the move the *starting colony* for this move. A *default located move* is a default move starting from the origin $(0, 0)$. ┘

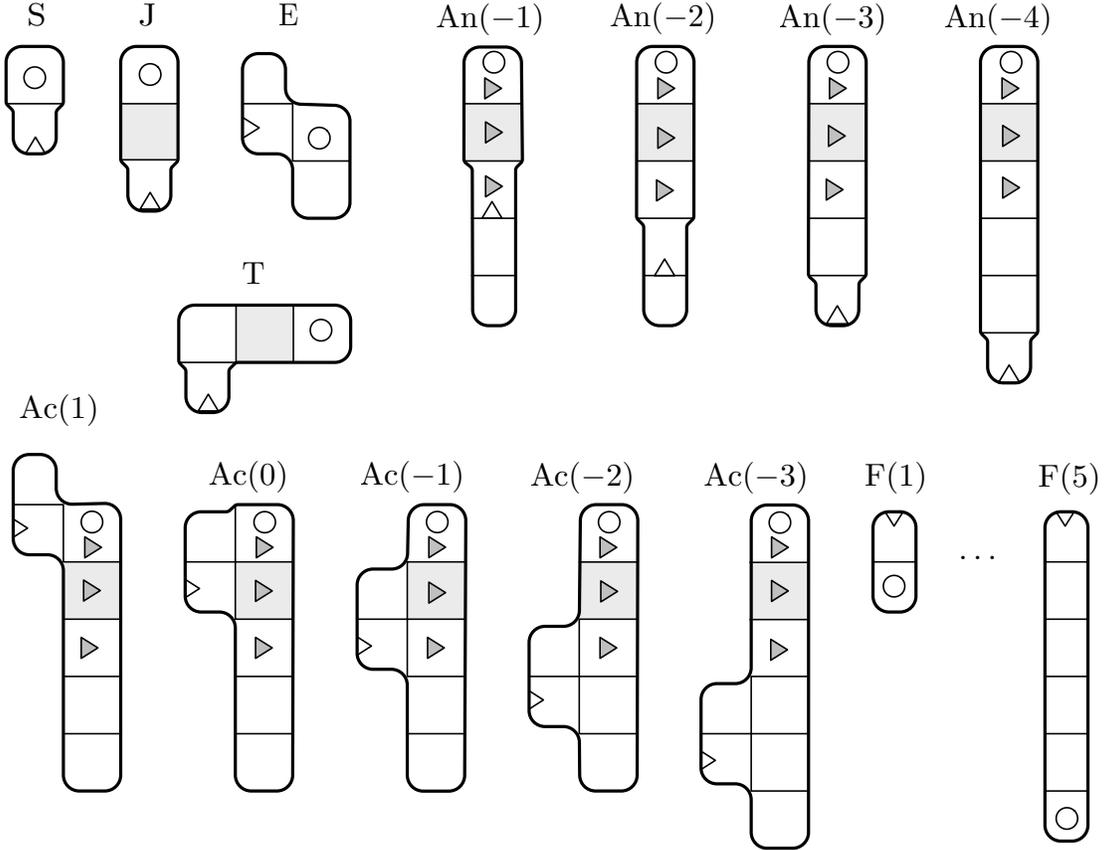


FIGURE 2. Possible moves. The template of each move is a union of squares. Grey squares are the obstacle positions of jumps and attacks. A white triangle marks the starting position. White circle marks the destination position, grey triangles the transit positions, showing the direction of the sweep. The level of a continuing attack is the difference between the height of the starting position and the height of the obstacle. Notation: S: step, J: jump, T: turn, E: escape, F: finish, A: attack, n: new, c: continuing, (i) : level or length i .

We will see that the devil's answer can end certain moves in two different ways: in "success" or in "failure". The intuitive meaning of failure is the failure to get through some "obstacle". A successful move always ends in its destination colony. A failed attack will end in one of its end colonies (possibly the destination colony). Formally, the definition of allowed moves, success and failure uses the notion of a point "clear" for starting a certain kind of move and a point "clear" for ending a certain kind of move in success or failure. These notions will be defined below.

Here is a description of the different kinds of move. They are also illustrated in Figure 2. These details will be motivated better once it is understood how they are used.

Definition 2.9. Moves are of the following kinds. Each move has a *starting direction* and *landing direction*. Only the turn move has a landing direction different from the starting direction, for all other moves, their *direction* is both the starting and landing direction.

Attacks and escapes also have a *passing direction*: so we talk about a northward attack or escape passing to the east.

We also distinguish *new* moves and *continuing* moves. The step, jump, turn and new attack are new moves, the continuing attack, escape and finish are continuing moves.

- A northward *step* has $\mathcal{H} = \{(0, 0), (0, 1)\}$, $\text{dest} = (0, 1)$.
- A northward *jump* has $\mathcal{H} = \{(0, 0), (0, 1), (0, 2)\}$, $\text{dest} = (0, 2)$. The position $(0, 1)$ of a northward jump is called its *obstacle position*. Other jumps are obtained if we rotate a northward step by multiples of 90 degrees.
- A *northward escape continuing to the east* has $\mathcal{H} = \{(0, 0), (0, 1), (1, 0), (1, -1)\}$, $\text{dest} = (1, 0)$. Other escapes are obtained by reflection and rotations.
- A *southward finish* of length $1 \leq i \leq 5$ has $\text{dest} = (0, -i)$, $\mathcal{H} = \{(0, 0), \dots, (0, -i)\}$. Other finishes are obtained by rotations.
- Let us describe the kind of moves called *northward attacks continuing to the east*. Other attacks are obtained using reflexion and rotations. It is more convenient to describe a *pre-template* \mathcal{H}' and a starting position \mathbf{p}' from which the real template \mathcal{H} is obtained by subtracting the starting position: $\mathcal{H} = \mathcal{H}' - \mathbf{p}'$. Of course, the starting position, destination position, and so on are also shifted when going from \mathcal{H}' to \mathcal{H} . The pre-template is a superset of

$$\mathcal{U} = \{(0, -3), (0, -2), (0, -1), (0, 0), (0, 1)\},$$

with the obstacle in $(0, 0)$, and the destination in $(0, 1)$.

All attacks have destination position $(0, 1)$ in \mathcal{H}' . They also have a set $\mathcal{T} \subset \mathcal{E}$ of *transit positions*: in \mathcal{H}' these are $(0, -1), (0, 0), (0, 1)$. Each attack has an integer *level* s with $-4 \leq s \leq 1$.

Attacks are also divided into *new* or *continuing* attacks. New attacks have level $-4 \leq s \leq -1$, the pre-template is equal to $\mathcal{U} \cup (0, s)$, and its start position is $\mathbf{p}' = (0, s)$. For continuing attacks, $-3 \leq s \leq 1$, $\mathcal{H}' = \mathcal{U} \cup \{(-1, s), (-1, s + 1), (0, -s)\}$, and the start position is $\mathbf{p}' = (-1, s)$.

- A northward *turn* is a combination of a northward step and an eastward or westward jump or new attack (the latter is northward-continuing). The direction of the jump or attack is called the *landing direction* of the turn.

The body M of a northward continuing attack or escape consists of left and right columns. The right column will be called the *reduced body* and denoted

$$\underline{M}.$$

┘

Definition 2.10. The definition of Δ_a and Δ_d uses the relations $\mathcal{K}_{\text{start}}$ and $\mathcal{K}_{\text{fail}}$. These are defined as follows, for measure μ . We have

$$\begin{aligned} \mathcal{K}_{\text{start}} &\subset \mathcal{M} \times \mathbb{Z}^2 \times P, \\ \mathcal{K}_{\text{fail}} &\subset \mathcal{M} \times \mathbb{Z}^2 \times \mathbb{Z}^2 \times \Pi. \end{aligned}$$

If $(\mu, \mathbf{p}, x) \in \mathcal{K}_{\text{start}}$ then point \mathbf{p} is *clear* for measure μ start a move in direction x . If $(\mu, \mathbf{p}, \mathbf{w}, z) \in \mathcal{K}_{\text{fail}}$ then point \mathbf{p} is *clear* for measure μ to end the located move (\mathbf{w}, z) in failure.

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Definition 2.11. Let us specify the set $\Delta_a(\mathbf{p}, \mu) \subset \Pi$ of possible moves for the angel when she is at position \mathbf{p} with measure μ .

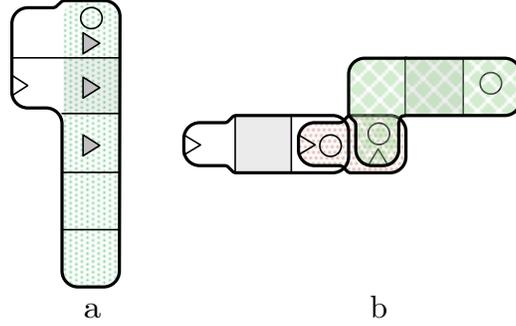


FIGURE 3. a: The reduced body of a continuing attack is shaded. b: An eastward jump followed by an eastward step followed by a northward-eastward turn.

We will only consider moves in the northward direction. The requirements for other directions are obtained using rotation and reflection.

- (a) A new move z with starting direction x is allowed only from a point \mathbf{p} that is clear in direction x , that is we must have $(\mu, \mathbf{p}, x) \in \mathcal{K}_{\text{start}}$.
A continuing move z is allowed from a point \mathbf{p} only if \mathbf{p} is clear for some failed located move (\mathbf{w}, z') having the same landing direction and passing direction: that is, $(\mu, \mathbf{p}, \mathbf{w}, z') \in \mathcal{K}_{\text{fail}}$.
- (b) The weight of the body is at most $3B$ (this bound is not important, just convenient).
- (c) The destination colony is (-1) -safe.
- (d) If the move is a step then the body is (-1) -safe.
- (e) If the move is a jump then the body is $1/2$ -good.
- (f) If the move is a northward escape then its reduced body (its right column) is (-1) -safe.
- (g) If the move is an attack then:
 - (A) The run in the reduced body below the obstacle colony is (-1) -safe.
 - (B) If it is a new attack, the body is good; if it is a continuing attack, the reduced body is good.
- (h) If the move is a turn then it satisfies the conditions of its constituent step and (jump or attack).

Let the *default move* be an eastward step. ┘

2.1.4. *Paths and the devil's constraints.* Before giving the devil's constraints, some more definitions are needed.

Definition 2.12. A *configuration* is a tuple

$$(t, \mathbf{p}_r, \mu, j).$$

Here, (t, \mathbf{p}, μ) is an essential configuration, and the symbol $j \in \{\text{succ}, \text{fail}\}$ shows whether the previous move of the angel *succeeded* or *failed*. The *default configuration* consists of the default configuration with $j = \text{succ}$ added. Let

$$\text{Locmoves}(B), \text{Configs}$$

be the sets of located moves and configurations respectively. A sequence $(\mathbf{a}_1, \dots, \mathbf{a}_m)$ of located moves with $\mathbf{a}_i = (\mathbf{w}_i, z_i)$ is called a *path* if $\mathbf{w}_{i+1} - \mathbf{w}_i \in \mathcal{E}(z_i)$ holds for all $i < m$. ┘

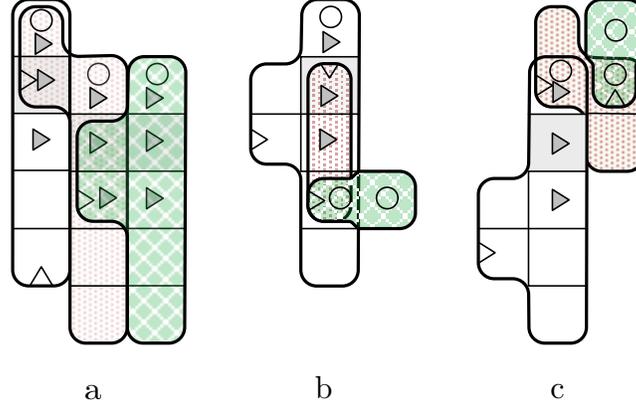


FIGURE 4. How attacks can be continued. a: A new attack fails on level 0 (on the level of its obstacle). Is followed by a continuing attack which picks up there (as shown by the first pair of grey and white triangles). This attack is of level 1 since the obstacle in the next column is one unit below the transit location where the failure occurred. The second attack fails on level -1 , and is followed by a continuing attack of level -1 since the next obstacle is on the same level as the previous one. b: A continuing northward attack fails on level 0, and is followed by a southward finish move and an eastward step. c: A continuing attack fails on level 1, and is followed by an escape move and a northward step.

Simple paths will be defined to be essentially self-nonintersecting. But the finish move and some steps following it in the same direction can overlap with the failed attack before it, so the following definition takes this into account.

Definition 2.13. Take a path, and let M_1, \dots, M_m be all the bodies of its located moves *except the finish moves*. Let S_i be the starting colony of the move of M_i . The path is *simple* if for all i we have $M_{i+1} \cap \bigcup_{j=1}^i M_j = S_{i+1}$. \lrcorner

Definition 2.14. If the angel's move is z_r , the set of possible moves of the devil is

$$\Delta_d = \Delta_d(\mathbf{p}_r, \mu_{t_r}, z_r) \subseteq \text{Configs}.$$

The devil's move is the next configuration $\mathbf{d}_{r+1} = (t_{r+1}, \mathbf{p}_{r+1}, \mu_{t_{r+1}}, j_{r+1})$, but his choice is restricted. First, of course $t_r \leq t_{r+1}$, further $0 \leq \mu_{t_{r+1}} - \mu_{t_r} \leq \sigma(t_{r+1} - t_r)$. The other restrictions defining Δ_d can be divided into *spatial* and *temporal* restrictions. Let us give the spatial restrictions first, assuming that the devil deposits the angel at a point \mathbf{p} with measure μ .

- (a) In case of success, \mathbf{p} is clear in the landing direction x of z , that is $(\mu, \mathbf{p}, x) \in \mathcal{K}_{\text{start}}$.
In case of failure the located move (\mathbf{w}, z) was a continuing move and \mathbf{p} is clear for failure for this move, that is $(\mu, \mathbf{p}, \mathbf{w}, z) \in \mathcal{K}_{\text{fail}}$.
- (b) If a new northward attack is not successful then the northward run of the body beginning from the starting colony is bad at its end time (of course, the same goes for other directions).
If a continuing attack is not successful then the reduced body is bad at its end time.

\lrcorner

Before giving the devil's temporal restrictions, some more notions concerning histories are needed.

Definition 2.15. Given a path $(\mathbf{a}_1, \dots, \mathbf{a}_m)$ with $\mathbf{a}_i = (\mathbf{w}_i, z_i)$, and a sequence $(\mathbf{d}_1, \dots, \mathbf{d}_{m+1})$ of configurations $\mathbf{d}_i = (t_i, \mathbf{p}_i, \mu_i, j_i)$, the sequence

$$(\mathbf{d}_1, \mathbf{a}_1, \mathbf{d}_2, \mathbf{a}_2, \dots, \mathbf{d}_m, \mathbf{a}_m, \dots, \mathbf{d}_{m+1})$$

will be called a *d-history* if its configurations obey the spatial restrictions and the restrictions on μ_t for the devil given above.

Under the same restrictions, the sequence $(\mathbf{d}_1, \mathbf{a}_1, \dots, \mathbf{d}_m, \mathbf{a}_m)$ is called an *a-history*. The set of all a-histories or d-histories will be denoted by

$$\text{Histories}_a(B), \text{Histories}_d(B).$$

The default d-history is (\mathbf{d}_0) , consisting of a single default configuration \mathbf{d}_0 . If χ is a history then let

$$\mathbf{a}(\chi)$$

be the path consisting of the angel's moves in it. ┘

Definition 2.16. We will use the addition notation, for example

$$(\mathbf{a}_1, \mathbf{d}_1, \dots, \mathbf{d}_m, \mathbf{a}_m) = (\mathbf{a}_1, \mathbf{d}_1, \dots, \mathbf{d}_m) + \mathbf{a}_m.$$

For another addition notation, if $\chi = (\mathbf{d}_1, \mathbf{a}_1, \dots, \mathbf{d}_i, \dots, \mathbf{a}_m, \mathbf{d}_{m+1})$ is a d-history then we can write $\chi = \chi_1 + \chi_2$ where $\chi_1 = (\mathbf{d}_1, \dots, \mathbf{a}_{i-1}, \mathbf{d}_i)$, $\chi_2 = (\mathbf{d}_i, \dots, \mathbf{a}_m, \mathbf{d}_{m+1})$ the d-histories from which it is composed. A d-history $(\mathbf{d}, \mathbf{a}, \mathbf{d}')$ will be called a *unit d-history*. Thus, every d-history is the sum of unit d-histories.

Similarly, if $\chi = (\mathbf{d}_1, \mathbf{a}_1, \dots, \mathbf{d}_i, \dots, \mathbf{a}_m)$ is an a-history then we can write $\chi = \chi_1 + \chi_2$ where $\chi_1 = (\mathbf{d}_1, \dots, \mathbf{a}_{i-1})$, $\chi_2 = (\mathbf{d}_i, \dots, \mathbf{a}_m)$, the a-histories from which it is composed. ┘

Definition 2.17. An 1-step a-history (\mathbf{d}, \mathbf{a}) will be called a *record*. Thus, every a-history is the sum of records. Let

$$\text{Records}(B) = \text{Configs} \times \text{Locmoves}(B)$$

denote the set of all possible records. The *default record* has the form $\alpha_0 = (\mathbf{d}_0, \mathbf{a}_0)$ consisting of the default configuration and the default move (eastward step). ┘

Definition 2.18. We define a *time bound* $\tau(\chi)$ for a history χ . Let μ be the measure in the last configuration, let U be the union of the bodies of all located moves in the path $\mathcal{P} = \mathbf{a}(\chi)$ and let n be the number of failed continuing attacks in χ . Then

$$\tau(\chi) = \rho_1 \mu(U) - \rho_2 n B + \tau_{\text{gc}}(\chi)$$

where $\tau_{\text{gc}}(\chi)$ is called the *geometric cost*, or the *geometric component* of the time bound, which we will define now. Let χ be a unit history containing a northward move that is not a turn, and let y_r and y_s be the y coordinates of the starting point and the endpoint respectively. We define the *geometric cost* of χ as $\tau_{\text{gc}}(\chi) = y_s - y_r$. (For an attack this can be negative.) For moves in other directions, the geometric cost is obtained by rotation accordingly. If χ is a single turn then $\tau_{\text{gc}}(\chi) = 8B$. If χ is an arbitrary history then let us decompose it into a sequence of unit histories χ_i and define $\tau_{\text{gc}}(\chi) = \sum_i \tau_{\text{gc}}(\chi_i)$.

Now, the *temporal restriction* of the devil is the following: the time of any d-history with a simple path is bounded by its time bound defined above.

Let us call an a-history or d-history *legal* if both the angel's and the devil's moves in it are permitted in the game, based on the sequence of preceding elements. ┘

Remarks 2.1.

1. Though the time bound contains negative terms, it never becomes negative because of $\rho_2 < \rho_1$: it can even be lowerbounded by a constant times the number of moves in the history.
2. Why do we “profit” only from failed continuing attacks and not from failed new attacks? This will be understood later, when we implement a scaled-up attack. A new attack typically needs some preparation steps that must be charged against its mass.

┘

In terms of histories, game \mathbb{G} can be described as follows. It is started from some initial configuration \mathbf{d}_1 , with $\mu = 0$, say in the middle of the cell $(0, 0)$. Now the angel adds her located move \mathbf{a}_1 to the history. The devil follows with the the next configuration \mathbf{d}_2 , and so on. So the angel’s moves can be viewed as located moves, the devil’s moves as configurations. Of course, each of these obeys the constraints given above.

Example 2.2. In order to specify a simple example of an AD-game we must specify the parameters Q, ξ, δ, ρ_i and the relations $\mathcal{K}_{\text{start}}, \mathcal{K}_{\text{fail}}$. Let $Q = 1$ and let us fix the other parameters in any way obeying the above restrictions (more restrictions come later to make scale-ups possible).

Let $(\mu, \mathbf{p}, z) \in \mathcal{K}_{\text{start}}$ if \mathbf{p} is (-1) -safe for μ . In case of a northward attack z let $(\mu, \mathbf{p}, z, \mathbf{w}) \in \mathcal{K}_{\text{fail}}$ if in the body of located move (\mathbf{w}, z) , the point \mathbf{p} is below the obstacle colony of the attack.

It is not hard to see that this game is essentially equivalent to the original angel-devil game.

┘

2.2. Scaling up. Let us define some concepts of cleanness for runs.

Definition 2.19. In a run U of colonies let us call the *obstacle* an element with largest weight (say the first one). A run will be called *i-step-clean* if every run of two consecutive colonies in U is *i*-safe. A run U of colonies of game \mathbb{G} is *i-unimodal* for some integer i if the runs on both sides of the obstacle are *i*-step-clean. It will be called *i-clean* if it is *i*-unimodal and every run of three consecutive colonies in U is $(i + 1)$ -good.

A run will be called *clean*, and so on if it is 0-clean, and so on.

Let U_1, \dots, U_n be a clean run, and let $1 = n_1 < n_2 < \dots < n_m = n$ be a sequence of indices with $n_{i+1} \leq n_i + 2$ such that U_{n_i} is safe, and if $n_{i+1} = n_i + 1$ then also $U_{n_i} \cup U_{n_{i+1}}$ is safe. This sequence will be called a *walk*: it consists of steps and jumps that can be carried out.

┘

Now we are ready to define scaled-up games.

Definition 2.20. Using the integer parameter Q introduced above, let

$$B^* = QB.$$

For clarity we will generally denote by α^* the elements of $\text{Records}(B^*)$, and use the $*$ notation similarly also in other instances where it cannot lead to confusion. Or, if α denotes an element of $\text{Records}(B^*)$ we might denote by α_* a record of $\text{Records}(B)$.

Each colony U^* of size B^* is the union of Q rows of colonies of size B , and also the union of Q columns of them. Let U^* consist of colonies U_{ij} ($1 \leq i, j \leq Q$) of size B . In the context of game \mathbb{G}^* the latter will be called *small colonies*, or *cells*.

Given a game \mathbb{G} of colony size B , the game

$$\mathbb{G}^* = \mathbb{G}(B^*, \Delta_a^*, \Delta_d^*)$$

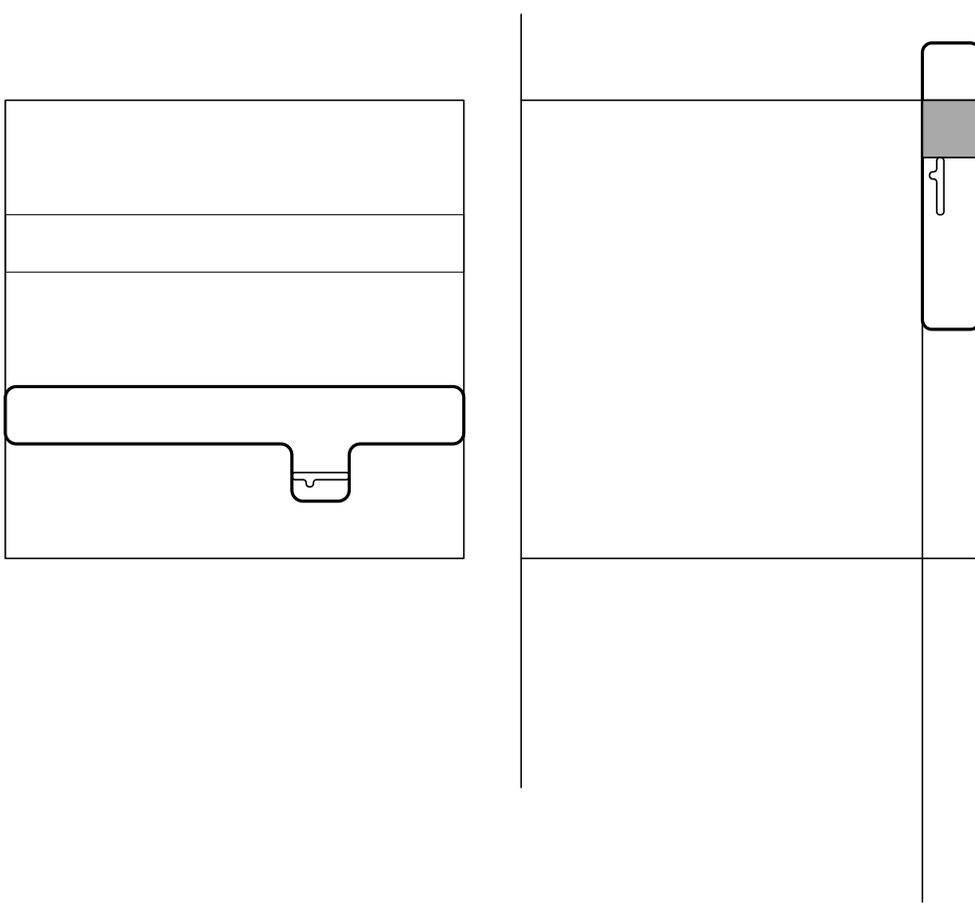


FIGURE 5. A northward clear point and a point that is clear for failure in a northward attack passing to the east.

will be defined similarly to game \mathbb{G} , but with colony size B^* , except that the sets $\mathcal{K}_{\text{start}}^*$, $\mathcal{K}_{\text{fail}}^*$ are defined as a function of the corresponding sets in \mathbb{G} , as given below. \lrcorner

Definition 2.21. Let a point p be in a cell U within the big colony U^* , and let μ be the current measure.

We will say that a point p is *clear* in direction x with respect to measure μ in game \mathbb{G}^* , that is $(\mu, p, x) \in \mathcal{K}_{\text{start}}^*$ if (assuming without loss of generality that $x = (0, 1)$, that is northward):

- (1) It is northward clear for μ in game \mathbb{G} , that is $(\mu, p, x) \in \mathcal{K}_{\text{start}}$.
- (2) Colony U^* is (-2) -safe.
- (3) There are at least $(\kappa - 2)$ clean rows in U^* north of U , and the first one is reachable from it in one (allowed) northwards step.

Let (w, z) be a located move with northward landing direction and eastward passing direction in \mathbb{G}^* such that U^* is one of the end colonies in the body M^* of (w, z) . We will say that a point p is clear for failure of (w, z) in \mathbb{G}^* if one of the following conditions is satisfied.

- (1) In the game \mathbb{G} , it is eastward clear (that is, $(\mu, \mathbf{p}, (1, 0)) \in \mathcal{K}_{\text{start}}$) and the column of cells in the body M^* south of the cell U (including U) is $(1/2)$ -step-clean.
- (2) There is a northward continuing located move (\mathbf{w}', z') passing to the east such that \mathbf{p} is clear for failure for (\mathbf{w}', z') , (that is $(\mu, \mathbf{p}, \mathbf{w}', z') \in \mathcal{K}_{\text{fail}}$), and the column of cells in the body M^* south of the cell U (not including U) is $(1/2)$ -step-clean.

┘

Intuitively, a point is northward clear if starting from it, we have some freedom to choose in which column to move northward. A point is clear for failure in a northward continuing attack passing to the east if it is on the east edge of the attack body, and a southward “retreat” is possible from it.

We are interested in translating moves of the angel in \mathbb{G}^* into sequences of moves in \mathbb{G} . Recall that a record is a one-step a-history.

Definition 2.22. Consider the pair of functions

$$\begin{aligned} \phi : \text{Histories}_d(B) \times \text{Records}(B^*) &\rightarrow \Pi \cup \{\text{Halt}\}, \\ (\chi, \mathbf{d}, \mathbf{a}) &\mapsto \phi(\chi \mid \mathbf{d}, \mathbf{a}), \\ J : \text{Histories}_d(B) \times \text{Records}(B^*) &\rightarrow \{\text{succ}, \text{fail}\}, \\ (\chi, \mathbf{d}, \mathbf{a}) &\mapsto J(\chi \mid \mathbf{d}, \mathbf{a}). \end{aligned}$$

To simplify writing if $(t', \mathbf{p}', \mu', j')$ is the last element of χ with $\mathbf{w}' = \lfloor \mathbf{p} \rfloor_B$ then we will write

$$\hat{\phi}(\chi \mid \mathbf{d}, \mathbf{a}) = (\mathbf{w}', \phi(\chi \mid \mathbf{d}, \mathbf{a})), \quad \hat{J}(\chi \mid \mathbf{d}, \mathbf{a}) = (t', \mathbf{p}', \mu', J(\chi \mid \mathbf{d}, \mathbf{a})).$$

We say that (ϕ, J) is an *implementation map* from game \mathbb{G} to game \mathbb{G}^* if it has the following properties, whenever χ is a legal d-history:

- (a) If $\phi(\chi \mid \mathbf{d}, \mathbf{a}) \neq \text{Halt}$ then it is a permitted move of the angel in game \mathbb{G} following any history ending in χ .
- (b) If $\phi(\chi \mid \mathbf{d}, \mathbf{a}) = \text{Halt}$ then χ consists of at least two steps, and $\hat{J}(\chi \mid \mathbf{d}, \mathbf{a})$ is a permitted move of the devil in the game \mathbb{G}^* following any history ending in (\mathbf{d}, \mathbf{a}) .

┘

The implementation can be viewed in the following way. The angel and devil of game \mathbb{G}^* play while pursuing a parallel game of \mathbb{G} as follows. When it is the angel’s turn in \mathbb{G}^* she will translate her move into a *local strategy* of \mathbb{G} , a strategy that can also halt. Step r of game \mathbb{G}^* corresponds to step s_r of game \mathbb{G} , but at the essential configuration as in game \mathbb{G}^* . A one-step d-history $(\mathbf{d}_r, \mathbf{a}_r, \mathbf{d}_{r+1})$ of game \mathbb{G}^* will correspond to a longer d-history

$$(\mathbf{d}_{*s_r}, \mathbf{a}_{*s_r}, \mathbf{d}_{*(s_r+1)}, \dots, \mathbf{a}_{*s_{r+1}-1}, \mathbf{d}_{*s_{r+1}})$$

of game \mathbb{G} generated as follows. Between steps s_r and s_{r+1} of game \mathbb{G} , the angel will use the following strategy. Let

$$\chi_{*i} = (\mathbf{d}_{*s_r}, \dots, \mathbf{a}_{*(i-1)} \mathbf{d}_{*i})$$

be the d-history in the game \mathbb{G} since the start of the implementation of the current move of \mathbb{G}^* , where $\mathbf{d}_{*i} = (t_{*i}, \mathbf{p}_{*i}, \mu_{*i}, j_{*i})$. Then the next move \mathbf{a}_{*i} in game \mathbb{G} is computed as follows, with $\alpha_r = (\mathbf{d}_r, \mathbf{a}_r)$:

$$\mathbf{a}_{*i} = \hat{\phi}(\chi_{*i} \mid \alpha_r).$$

From here, the local d-history is extended as $\chi'_* = \chi_{*i} + \mathbf{a}_{*i}$. The devil of \mathbb{G}^* allows the devil of \mathbb{G} to play and generate the d-history $\chi_{*i+1} = \chi'_* + \mathbf{d}_{*(i+1)}$. Step number s_{r+1} will

be reached in \mathbb{G} when the angel in game \mathbb{G} chooses the symbol Halt. At this point the devil of game \mathbb{G}^* chooses the new configuration of game \mathbb{G}^* , namely

$$\mathbf{d}_{r+1} = \hat{J}(\chi_* \mid \alpha_r).$$

This next configuration of game \mathbb{G}^* is almost the same as the last configuration of the subgame: the essential configurations are the same, only the question whether the last move of \mathbb{G}^* (the one we have just implemented) is successful will be decided using the function $J(\cdot)$.

Thus to any d-history χ of \mathbb{G}^* corresponds some d-history χ_* of game \mathbb{G} that shares with χ the essential initial and final configurations. The correspondence assigns disjoint subhistories of χ_* to each unit history of χ . Of course, χ_* is not a function of χ but the map (ϕ, J) translates any strategy that the angel has for \mathbb{G}^* into a strategy for \mathbb{G} .

Definition 2.23. An *amplifier* consists of a sequence of games $\mathbb{G}_1, \mathbb{G}_2, \dots$ where $\mathbb{G}_{k+1} = \mathbb{G}_k^*$ and implementation maps (ϕ_k, J_k) from \mathbb{G}_k to \mathbb{G}_{k+1} . \lrcorner

In the amplifier built in the present paper the maps (ϕ_k, J_k) will not depend on k in any significant way: only the scale changes.

2.3. The main lemma. Before stating the main lemma, from which the theorem will follow easily, let us constrain our parameters.

Let

$$\nu = 17Q, \quad \kappa \geq 12 \tag{3}$$

be an integer parameters. The parameter ν will serve as the upper bound on the number of small moves in an implementation of a big move. The parameter κ will have the approximate role that in a safe colony there will be at least κ guaranteed good rows. Below, the parameter θ has the meaning that the time taken by a single move will be upperbounded by θB . Let

$$Q > 2\kappa/(1 - \xi), \tag{4}$$

$$\rho_2 = 8, \tag{5}$$

$$\rho_1 > 22Q/(1 - \xi), \tag{6}$$

$$\theta = 2(6 + 3\rho_1), \tag{7}$$

$$\delta < \min((1 - \xi)/6, (\xi - 2/3)/Q), \tag{8}$$

$$\sigma < \min(\delta/(3\nu\theta), 1/(2\rho_1)). \tag{9}$$

These inequalities can be satisfied by first choosing Q to satisfy (4), then ρ_1 to satisfy (6), then choosing δ to satisfy the first inequality of (8) and finally choosing σ to satisfy (9).

Remark 2.3. We made no attempt to optimize the parameters. Not fixing them, only constraining them with inequalities has only the purpose to leave the “machinery” somewhat open to later adjustments. Considering Q as a variable, these relations imply $\sigma \sim Q^{-2}$. With more careful analysis one could certainly achieve $\sigma \sim Q^{-1}$. \lrcorner

Lemma 2.4 (Main). *If our parameters satisfy the above relations then there is an implementation (ϕ, J) .*

We prove this lemma in the following sections, and then apply it to prove the theorem.

3. RELATIONS AMONG PARAMETERS

The following simple lemma illustrates the use of cleanness.

Lemma 3.1. *If a run is clean then between any two safe points of it there is a walk.*

Proof. A clean run contains at most one unsafe colony, the obstacle. Suppose it is U_k with $1 < k < n$. Then we can make $n_i = i$ for all $i < k$, then $n_i = i + 1$ for $k \leq i < n$. \square

Lemma 3.2. *The following relations hold for our thresholds.*

- (a) *The time passed during any move is at most θB .*
 (b) *If an attack has a 1-good reduced body then it succeeds.*

Proof. Let us prove (a). Suppose that the move takes time x and has body M . Let μ_0 be the measure before the move and μ_1 after it. By the requirements, $\mu_0(M) < 3B$. Note that $\tau_{\text{gc}}(\chi) < 6B$ for the history χ consisting of the single move in question. Now therefore we have

$$\begin{aligned} x &\leq \rho_1 \mu_1(M) + \tau_{\text{gc}}(\chi) \leq \rho_1(\mu_0(M) + \sigma x) + 6B \leq \rho_1(3B + \sigma x) + 6B, \\ x &\leq \frac{B(6 + 3\rho_1)}{1 - \rho_1\sigma} \leq 2B(6 + 3\rho_1) = B\theta, \end{aligned}$$

using (9) and (7).

To prove (b) note that a 1-good run is still good after the move due to $\delta > \theta\sigma$, and the attack could fail only if this run became bad. \square

A good enough run is clean, as the lemma below shows.

Lemma 3.3. *If a run is $(-Q)$ -good then it is 1-unimodal. Consequently, if it is $(i + 1)$ -good then it is i -clean for $0 \leq i \leq 1$.*

Proof. Let a be the weight of the obstacle and b, c two weights of non-obstacle colonies. Then we have

$$b + c \leq (2/3)(a + b + c) \leq (2/3)(1 + Q\delta)B < (\xi - \delta)B$$

by (8). The ‘‘consequently’’ part follows immediately from the definition of i -cleanness. \square

Lemma 3.4. *Suppose that for rectangle U consisting of Q horizontal runs of colonies below each other, we have $\mu(U) \leq QB(\xi + \delta) + B$. Then at least κ of the horizontal runs are 1-clean.*

In particular, if a colony of \mathbb{G}^ is (-1) -safe then at least κ of its rows are 1-clean.*

Proof. Suppose that U does not have κ rows that are 2-good (and thus 1-clean). Then

$$\begin{aligned} (Q - (\kappa - 1))B(1 - 2\delta) &< \mu(U) < QB(\xi + \delta) + B, \\ 1 - \xi - \kappa/Q &< \delta(3 + 2(\kappa - 1)/Q), \\ (1 - \xi)/2 &< \delta(4 - \xi) \end{aligned} \tag{4}$$

contradicting (8). \square

4. THE IMPLEMENTATION MAP

This section proves the main lemma.

Let μ_0 be the measure at the beginning of the big move, and p_0 the initial point. Unless saying otherwise, the properties of parts of M^* are understood with respect to μ_0 . We will make most decisions based on the measure μ_0 . Our map will implement each big move using at most ν small moves, where ν was defined in (3). Due to Lemma 3.2 and

relation (9) this will imply that, for example, a run required to be safe with respect to μ_0 will still be (-1) -safe at the end. Let us make this statement explicit:

Lemma 4.1. *In any sequence of $\leq \nu$ moves, the total mass increases by less than δB .*

This extra tolerance in the initial requirements insures that any planned steps, jumps and turns remain executable by the time we actually arrive at the point of executing them. With attacks this is not the case, they can fail or we may find immediately before executing an attack that it is not executable anymore since the reduced body is not good anymore.

Definition 4.1. In an implementation, colonies of \mathbb{G}^* will be called *large* colonies, and colonies of \mathbb{G} *small* colonies, or *cells*. Moves in the game \mathbb{G}^* will be called *big* moves, and moves in the game \mathbb{G} *small* moves, or simply *moves*. A big move has starting colony and body S^* , M^* . When it has a destination colony that will be denoted by D^* . \square

4.1. **Plan for estimating the time.** Let

$$\chi = (\mathbf{d}_1, \mathbf{a}_1, \dots, \mathbf{d}_{m+1})$$

be a history of \mathbb{G}^* . As shown in Subsection 2.2, in the implementation there corresponds to χ a history χ_* of game \mathbb{G} , which shares the initial and final essential configurations of χ . Segment $(\mathbf{d}_r, \mathbf{a}_r, \mathbf{d}_{r+1})$ of χ corresponds to segment

$$(\mathbf{d}_{*s_r}, \mathbf{a}_{*s_r}, \mathbf{d}_{*(s_r+1)}, \dots, \mathbf{a}_{*s_{r+1}-1}, \mathbf{d}_{*s_{r+1}})$$

of χ_* . If M_r^* is the body of located move \mathbf{a}_r and M_i is the body of located move \mathbf{a}_{*i} then our implementation will give $M_i \subset M_r^*$ for all $s_r \leq i \leq s_{r+1} - 1$. So the body of the path of the implementation of each move is in the body of the implemented move.

If the path $\mathbf{a}(\chi)$ of history χ is simple then we will implement it via a simple path $\mathbf{a}(\chi_*)$. (This goal accounts for some of the complexity of the implementation.)

Since \mathbb{G} is an AD-game, we can estimate the time of path χ_* by the time bound introduced in Definition 2.18. We will then show that this estimate obeys the time bound required by game \mathbb{G}^* . Let μ be the measure in the last configuration, let U and U_* be the union of the bodies of all located moves in the path $\mathbf{a}(\chi)$ and $\mathbf{a}(\chi_*)$ respectively. Let n be the number of of failed continuing attacks in χ and n_* be the number of of failed attacks in χ_* . Then by the time bound of game \mathbb{G} we have

$$\tau(\chi_*) = \rho_1 \mu(U_*) - \rho_2 n_* B + \tau_{\text{gc}}(\chi_*).$$

Our goal is to show that this expression is bounded above by

$$\tau(\chi) = \rho_1 \mu(U) - \rho_2 n Q B + \tau_{\text{gc}}(\chi).$$

Ignoring the negative terms first, as we noted $U_* \subset U$, so of course we have $\mu(U_*) \leq \mu(U)$. But $\tau_{\text{gc}}(\chi_*)$ will typically be larger than $\tau_{\text{gc}}(\chi)$.

If $\chi = \chi_1 + \dots + \chi_m$ where χ_m are unit histories and χ_{*i} is the segment of χ_* corresponding to χ_i then $\tau_{\text{gc}}(\chi) = \sum_i \tau_{\text{gc}}(\chi_i)$, and $\tau_{\text{gc}}(\chi_*) = \sum_i \tau_{\text{gc}}(\chi_{*i})$. Trying to bound each $\tau_{\text{gc}}(\chi_{*i})$ by the geometric cost $\tau_{\text{gc}}(\chi_i)$ of the big move from which it was “translated”, we will frequently have $\tau_{\text{gc}}(\chi_{*i}) > \tau_{\text{gc}}(\chi_i)$. Let us call the difference $\tau_{\text{gc}}(\chi_{*i}) - \tau_{\text{gc}}(\chi_i)$ the *extra geometric cost*. The basic strategy in the implementation is to “charge” the extra geometric cost to the weight of some sets in the difference $U \setminus U_*$. This suffices for the translation of a big step.

Unfortunately in the implementation of the other moves χ_i , there may not be enough mass outside U_* . We will compensate the geometric cost by the negative contribution $\rho_2 B$ of some failed continuing attacks in the implementation. Of course if χ_i is a failed continuing attack itself then this cannot be done, fortunately then it need not be.

Definition 4.2. The value $\rho_2 B$ will be called the *profit* of any failed continuing attack. \lrcorner

4.2. General properties.

Definition 4.3. Let $U = U_1 \cup \dots \cup U_n$ be a vertical run of colonies. We will say that U_i is *secure* in U provided $U_{i-1} \cup U_i$ is safe (if $i > 1$) and $U_i \cup U_{i+1}$ is safe (if $i < n$).

We will say that a horizontal and vertical run intersect *securely* if the intersection colony is secure either in the horizontal run or in the vertical run. \lrcorner

Lemma 4.2. *Let R be a clean row in a rectangle. Then there are at most 3 clean columns that do not intersect R securely.*

Proof. Indeed any clean column that intersects R in a position different from the obstacle and its neighbors intersects R securely. \square

Remark 4.3. In the procedure below, when a row and a column intersect securely we will sometimes say that we first walk in the row and then continue walking in the column. But it is understood that if the move before the intersection is a step and the one after the intersection is a jump then these two moves are actually replaced by a single turn move. \lrcorner

The destination colony of a big move is (-1) -safe and therefore due to Lemma 3.4 has at least κ rows that are 1-clean.

Definition 4.4. Let $C(0)$ be the starting column of the angel.

When starting from a northward-clear point in S^* , we will denote by $R'(0)$ the row to which it is possible to step north. For $i \geq 1$ let $R'(i)$ denote the $(i + 1)$ th clean row of S^* starting from the south.

Let $R''(i)$ denote the i th clean row of D^* starting from the south with the additional property $R''(i) > 1$. \lrcorner

In the implementation of a big step, jump or new attack, ideally we would just walk in column $C(0)$ to a cell below row $R''(1)$ from which it is reachable in one step. We will do something else only if this is not possible.

Definition 4.5. Certain runs of cells in the body of each implemented big move will be called *scapegoat runs*. Consider a history χ of \mathbb{G}^* and its implementation χ_* in \mathbb{G} . We will make sure that all scapegoat runs will be disjoint of each other as well as of the body of χ_* .

Certain subhistories will be called *digressions*. Each digression will be *charged* to some scapegoat run, and different digressions will be charged to different scapegoat runs. \lrcorner

Definition 4.6. We will say that row R is *securely reachable* from a cell U below it if the upward vertical run from U to the last cell U' below R is clean and the step from U' to R is safe.

Let us call the *blameable run* of U, R the vertical run starting from the cell above U and ending in R . \lrcorner

Let us lowerbound the weight of a blameable run.

Lemma 4.4. *Suppose that R is not securely reachable from U . Let V be the blameable run of U, R , and μ the current measure. Then we have*

$$\mu(V) \geq 0.5B(1 - \xi).$$

Proof. If the step with body $U' \cup U''$ from cell U' below R to cell U'' in R is not safe then $\mu(U' \cup U'') \geq \xi B$. Suppose it is safe. Since the run V' from U to U' is not clean, it follows

from Lemma 3.3 that it is not 1-good. Using the fact that the angel's current position is in a (-1) -safe cell,

$$\mu(V') \geq B(1 - \delta), \quad (10)$$

$$\mu(U) \leq B(\xi + \delta). \quad (11)$$

$$\mu(V' \setminus U) \geq B(1 - \xi - 2\delta) \geq 0.5B(1 - \xi) \quad (12)$$

due to (8). \square

So a blameable run has weight $\geq 0.5B(1 - \xi)$. If there is extra geometric cost (at most QB for some constant c), then the inequalities in Subsection 2.3 show that we will be able to charge it all against ρ_1 times this weight, the weight of an appropriate scapegoat run.

(This is crude, a factor of Q is lost here unnecessarily.)

4.3. Big step. Let us define now the translation of a big northward step.

1. Let us call the starting cell U . If some $R''(i)$ is securely reachable from $R'(0) \cap C(0)$ then let $R''(i''_0)$ be the first such (note that $R''(i''_0) > 1$). We will walk to the colony below $R''(i''_0)$ in $C(0)$. Row $R''(i''_0)$ will still be clean, so we will be done.

Suppose now that $R''(1)$ is not securely reachable from $R'(0) \cap C(0)$. Since the body of the big step is (-1) -safe, Lemma 3.4 implies that it has at least κ columns that are clean.

Let $C(1)$ be one that is securely reachable from $R'(0) \cap C(0)$. Let i''_1 be the first $i > 1$ such that $R''(i)$ is securely reachable from $R'(0) \cap C(1)$. We step up to $R'(0)$, then walk to $R'(0) \cap C(1)$ and then to the cell below $R''(i''_1) \cap C(1)$. The scapegoat is the blameable run of $R'(0) \cap C(0)$, $R''(1)$.

Remark 4.5. We choose $i > 1$ in order to be above $R''(1)$, since a cell of $R''(1)$ may be part of a scapegoat run, and the clean row under which we will end up should be disjoint from the scapegoat run, in order to be avoid intersecting the scapegoat run in the implementation of the next big move. In what follows we choose larger and larger i'' values for similar reasons. \lrcorner

The total geometric cost is at most $B(16 + Q + 2Q) \leq 4QB$ (with two turns and two straight runs), and the total number of moves is at most $3Q$.

2. Suppose that the starting point is not northward clear, but it is eastward clean, let $C(1)$ be the clean column one step to the right of the current cell U .

Again, if some $R''(i)$ is securely reachable from U then let $R''(i''_0)$ be the first such. We will walk up to the colony below $R''(i''_0)$.

If we do not have this case (see Figure 6) then since the body of the big step is (-1) -safe, again there is a clean column $C(2)$ of M^* , to the east of $C(1)$. Let i''_1 be the first $i > 1$ such that $R''(i)$ securely intersects $C(2)$. There is a clean row $R'(1)$ of S^* securely intersecting both $C(1)$ and $C(0)$. We will step into $C(1)$, then walk to $R'(1) \cap C(1)$, then to $R'(1) \cap C(2)$, finally walk in $C(2)$ to the cell below $R''(i''_1)$. The scapegoat is the blameable run of U , $R''(1)$.

The total geometric cost is at most $B(24 + Q + Q + 2Q) \leq 5QB$, adding up the cost of at most three turns and three runs, and the total number of moves is at most $4Q$.

4.4. Big jump. Consider a big northward jump, starting from a northward clear point. There is a clean row $R'(0)$ to which it is possible to step north from the current cell U . We will use Definition 4.4 again.

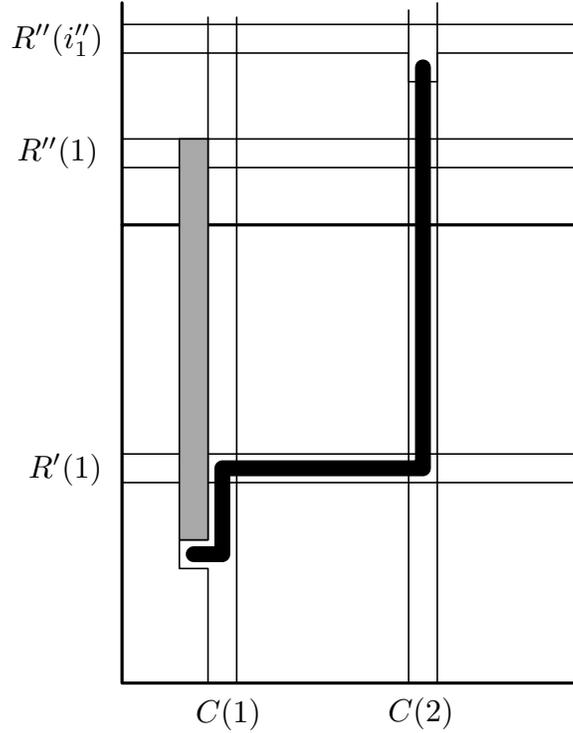


FIGURE 6. Implementing a northward step from an eastward-clear point. The grey rectangle is the scapegoat.

4.4.1. *If a clean column exists.* Suppose that M^* has a clean column $C(1)$. Then we will proceed somewhat as in part 1 of the implementation of a big step, with possibly yet another digression.

If some $R''(i)$ is securely reachable from $R'(0) \cap C(0)$ then let $R''(i''_0)$ be the first such. We will walk to the colony below $R''(i''_0)$ in $C(0)$. Row $R''(i''_0)$ will still be clean, so we will be done.

Suppose now that $R''(1)$ is not securely reachable from $R'(0) \cap C(0)$. If $C(1)$ was securely reachable from $R'(1) \cap C(0)$ then we could proceed just as in the implementation of a big step, but this is not guaranteed now, so suppose it is not so. Let $R'(i'_1)$ be the lowest clean row of S^* above $R'(0)$ securely intersecting $C(1)$ and let $C(2)$ be a column whose intersection colony is secure both with $R'(0)$ and $R'(i'_1)$. We step up to $R'(0)$, then walk in $R'(0)$ to $C(2)$, further in $C(2)$ to $R'(i_1)$.

If some $R''(i)$ for $i > 1$ is securely reachable from $R'(i'_1) \cap C(2)$ then let $R''(i''_2)$ be the first such. We will walk to the colony below $R''(i''_2)$ in $C(2)$, charging everything to the scapegoat run of $U, R''(1)$. Otherwise let i''_2 be the first $i > 1$ such that $R''(i)$ is securely reachable from $R'(i'_1) \cap C(1)$. We walk in $R'(i'_1)$ to $C(1)$, and then $C(1)$ to the cell below $R''(i''_2)$. The whole operation will be charged to the scapegoat run of $R''(i'_1) \cap C(2), R''(i'_1)$,

The geometric cost in this worst case is at most $B(32 + 3Q + 3Q) \leq 7QB$, the number of moves is at most $6Q$. We will be able to charge the geometric cost to a single scapegoat run due to (6).

From now on we suppose that no clean column exists in M^* .

4.4.2. *Obstacles.* Let us draw some consequences of the fact that the body M^* is good. (For a big jump we actually have 1-goodness, but the analysis will also be applied to the implementation of attacks.)

Lemma 4.6. *If M^* has no 1-good column then every column is unimodal.*

Proof. Suppose that no column is 1-good. Let $w_1 \leq w_2 \leq \dots \leq w_Q$ be the weights of all the columns, ordered, so $w_i \geq (1 - \delta)B$ for all i . Then,

$$\begin{aligned} QB &\geq w_1 + \dots + w_Q \geq (Q - 1)B(1 - \delta) + w_Q, \\ B(1 + (Q - 1)\delta) &\geq w_Q. \end{aligned}$$

Now unimodality is implied by Lemma 3.3. \square

So now we know that each column is unimodal. On the other hand, if we still manage to pass through in a simple way then there plenty of ways to charge it, since all columns of M^* are heavy.

Suppose that for some $i \geq 1$ there is a column $C(1)$ of M^* whose obstacle is below $R'(i)$. It is easy to perform an implementation that gets us to $R'(i)$ via some column $C(2)$ that is clean in S^* and intersects $R'(0)$ and $R'(i)$ securely, then from there to $C(1)$, and finally walks north on $C(1)$ to an appropriate row $R''(i')$. Charging is done like in earlier similar cases (but as mentioned above is not a problem anyway). The extra geometric cost and the number of moves have also the same bounds. The case remains that no obstacles are below any $R'(i)$.

Suppose that there is some obstacle above $R''(1)$, in some column $C(1)$. Then we can walk to $C(1)$ and then in $C(1)$ to below $R''(1)$, using again possibly an intermediate row $R'(i')$ and column $C(2)$. Charging and bounding the extra geometric cost and number of moves is done as before. The case remains that there are no obstacles below $R'(i)$ for any i and above $R''(1)$.

Let us make another observation about obstacles.

Lemma 4.7. *Suppose that there are i, j such that $r_j, r_{j+1} \notin \{i, i+1\}$. Then there is a $q \in \{0, 1\}$ such that the horizontal run $M_{i+q,j} \cup M_{i+q,j+1}$ is safe. In other words, it is possible to pass horizontally between the two obstacles.*

Proof. Since the runs $M_{i+1,j} \cup M_{i+2,j}$ and $M_{i+1,j+1} \cup M_{i+2,j+1}$ are safe, the measure of the union of these four cells is $< 2B\xi$, hence one of the horizontal runs $M_{i+q,j} \cup M_{i+q,j+1}$ with $q \in \{1, 2\}$ has measure $< B\xi$. \square

Consider the case when there is a j with $|r_j - r_{j+1}| > 2$. We can escape then as follows. Assume without loss of generality $r_{j+1} < r_j - 2$. Then by the lemma there is a $q \in \{1, 2\}$ such that the run $M_{r_j-q,j} \cup M_{r_j-q,j+1}$ is safe. Therefore one can walk to column j (again possibly using an intermediate column $C(2)$ and row $R'(i')$), then up to $(r_j - q, j)$, pass to $(r_j - q, j + 1)$ and further up in column $(j + 1)$. The geometric cost is at most $B(40 + 2Q + 3Q) \leq 6QB$, the number of moves is at most $5Q$. Charging is done as usual, but as remarked above is not a problem anyway.

Definition 4.7. We will say that in a big jump, the body M^* , or in a big attack, the reduced body \underline{M}^* has the *marginal case* if the following holds:

- (a) Every column is unimodal.
- (b) The rows $R'(i)$ contain no obstacle.
- (c) For all j we have $|r_j - r_{j+1}| \leq 1$.

Otherwise we have the *straight case*. \lrcorner

A success branch starts after a successful move. If it was a successful attack we escape, at no extra geometric cost. Otherwise we are either before the initial attack in column 1, or are coming from the left.

We are in some column j , in some current row i_0 , with $r_j - 5 \leq i_0 < r_j$. If a new attack is allowed (namely the body of that attack is good), then we make it, of level $i_0 - r_j$. We end up at a height i_1 with $|i_1 - r_{j+1}| \leq 1$. If it is not allowed then if $j = Q$ we halt, otherwise what follows will be called an *evasion*.

By assumption $r_j - 2 \leq r_{j+1} \leq r_j + 2$. If $r_{j+1} < r_j$ then we step up or down from i_0 to $r_{j+1} - q$ for some $q \in \{1, 2\}$ and pass to the right. If $r_{j+1} \geq r_j$ then we step up or down from i_0 to $r_j - q$ for some $q \in \{1, 2\}$ and pass to the right. (Note that we never have to pass down if we are in column 1, since there we are at height $\min(r_1 - 2, r_2 - 2)$.) In both cases we end up at a new height i_1 with $r_{j+1} - 4 \leq i_1 < r_{j+1}$.

An extra $\rho_2 B$ will also be charged to that scapegoat cell, to compensate for the profit that we do not have since we did not have a continuing attack. Inequalities (5) and (6) show that the scapegoat cell has enough weight for these charges. (This part is not important for the case of a big step, but we will reuse the analysis of the marginal case of the big step in the case of a failed attack, where the profit is needed.)

A success branch spends at most 5 moves in each column (5 in case of evasion, 1 in case of new attack). If we made a new attack but we came from an evasion, we charge the geometric cost of the new attack and the missing profit to the scapegoat cell above the northwards bad position in the previous evasion. Inequalities (5) and (6) show that the cell has enough weight for these charges even in addition to the charges made for the evasion itself.

4.4.5. *Failure branch.* A failure branch starts after a failed attack, so $|i_0 - r_j| \leq 1$. Let i_1 be height of the bottom cell of the body of that attack. We halt if $j = Q$ (never happens in the implementation of a big step). If $i_0 - r_{j+1} > 1$ then we make an escape move and then escape, at no extra geometric cost. Suppose $i_0 - r_{j+1} \leq 1$. Since $r_{j+1} \leq r_j + 2$ we have $-3 \leq i_0 - r_{j+1} \leq 1$. If a continuing attack with obstacle r_{j+1} is allowed we make it, at no extra geometric cost or lost profit. Suppose it is not allowed. Let

$$i_2 = \min(i_0 - 1, r_j - 1, r_{j+1} - 1).$$

By Lemma 4.7 a step right is possible at height $i_2 - q$ for some $q \in \{0, 1\}$. We can get to $i_2 - q$ using a finish move taking us to $\max(i_1, i_2 - q)$ and following it with some downward steps. After moving to column $(j + 1)$ we are positioned for a success branch, at a position $(i_3, j + 1)$ with $r_{j+1} - 4 \leq i_3 < r_{j+1}$.

A failure branch spends at most 6 moves per column, at a geometric cost of at most $7B$ which we will charge to the scapegoat cell of the disallowed attack. We will even charge the next evasion to it, which will still be allowed by (6).

The final escape has no geometric cost and takes at most $3Q$ moves.

4.4.6. *Summary of costs of the marginal case of a big step.* In the marginal case the extra geometric cost is at most $B(5 + 4Q) \leq 5QB$ for the part before the sweep (the one during the sweep is accounted for). The number of moves is at most $4Q + 7Q + 3Q = 14Q$, where we also counted the moves of the final escape.

If we succeed before column C_Q then we can charge the extra geometric cost to the blameable run from $R'(0) \cap C_Q$ to $R''(1)$. Otherwise we turn a profit in all columns but possibly the first one (not a continuing attack) and the last one (not a failed attack), so we can charge the extra geometric cost to this profit of size $(Q - 2)\rho_2 B$, via (5).

4.5. Big attack.

4.5.1. *New attack.* We only look at northward attacks passing to the right. A new northward attack is implemented just like a jump: the only difference is that it may actually fail. We say that the attack fails if either the last attack in column Q fails or we find ourselves in an northwards bad position in the last column. *This is the definition of the function $J(\cdot)$ of the implementation.*

Also, row $R'(i)$ is defined, instead of Definition 4.4, as the $(i + 1)$ th southernmost 1-clean row of the colony immediately below the obstacle colony of the big attack. Then we can conclude that in the marginal case all obstacles are above even these rows $R'(i)$.

The bounds on the geometric cost change only by the consideration the body of an attack may be by $3Q$ longer than that of a jump. So the extra geometric cost will be bounded by $B(8 + 7Q) \leq 8QB$ and the number of moves by $17Q$.

Suppose therefore that our attack with body M_1^* is a continuing one, then the previous move was a failed attack, also to the north and passing to the right, with some body M_0^* . The last move of its implementation was either a failed attack, or a successful move ending in an northwards bad position. We are in the rightmost column of M_0^* .

4.5.2. *Marginal case.* If the attack of M_1^* has the marginal case, as defined in Definition 4.7 then its columns are unimodal. In this case we can just continue the sweep to the right into M_1^* seamlessly, except that in the first column of M_1^* we may have to walk up as in case of the first column of the marginal case of a big jump. (One could also just step back to the last column of M_0^* and escape, but we will pass this opportunity, for the sake of orderliness.)

The walk-up entails no extra geometric cost. The cost of the rest of the implementation will be estimated just as for new attacks. If the attack is a failed one we always have the marginal case, and we do not escape. In this case each column contributes its required profit: the ones that did not were compensated by others, as shown in the implementation of the marginal case of a big step.

4.5.3. *Straight case.* Assume now that the the attack of M_1^* has the straight case, and we are at position (m, Q) of M_0^* .

1. Suppose that the northwards run of column 1 of M_1^* starting from position $(m - 1)$ to row $R''(1)$ is safe. If the last step of the big attack of M_0^* was a failed attack we make an escape move into m and then escape, at no extra geometric cost.

If the last step was a successful move then the starting position is northwards bad. We step right if this is possible, otherwise we step down and step right: just as in the evasion procedure, this is always possible. Then we escape north. The geometric cost of these digression steps is charged to the scapegoat cell above the northwards bad starting position. (Note that the body of the big escape and of each big continuing attack contains a big colony above the starting big colony, so the scapegoat cell is inside this body.)

2. Suppose now that the northwards run of column 1 of M_1^* starting from position $(m - 1)$ is not safe. In case the last step was a failed attack we add a southward finish move taking us to the bottom of the reduced body of that attack. Now we are in a southward-clear point in a step-clean run below the obstacle of the last column of M_0^* , which stretches down at least two big colonies below the current big colony. One of these big colonies, say U_0^* , is such that its right neighbor U_1^* is below the obstacle of the big attack of M_1^* .

Let V^* be the set consisting of U_1^* and the last column of U_0^* . This set has at least κ clean rows. Indeed, since U_1^* is safe, we have $\mu(U_1^*) \leq \xi BQ$. For the last column C_Q of

U_0^* we have $\mu(C_Q) \leq (1 + Q\delta)B$. So we have

$$\mu(V^*) \leq BQ\xi + B + BQ\delta,$$

therefore Lemma 3.4 is applicable.

- a. Suppose that there is a column $C(1)$ of \underline{M}^* different from the first column, a clean row $R(1)$ of V^* that intersects both C_Q and $C(1)$ securely, and an $i_1'' > 1$ such that $R''(i_1'')$ is securely reachable in $C(1)$ from $R(1)$. (This is always the case if \underline{M}_1^* has a 1-good column different from the first one.) We then walk down into $R(1)$, then walk over to $C(1)$, and finally walk up in $C(1)$ to under $R''(i_1'')$. The geometric cost is at most $B(16 + 2Q + Q + 6Q) = B(16 + 9Q) \leq 10QB$, and the number of moves is at most $9Q$. We charge all this to the unsafe run from $(m - 1, 1)$ to $R''(1)$.
- b. Suppose now that the above case does not hold. Let $R(1)$ be any clean row of \underline{M}_1^* securely intersecting C_Q , and let $C(1)$ be any column of \underline{M}_1^* different from the first column, securely intersecting $R(1)$, then $R''(2)$ is not securely reachable from $R(1) \cap C(1)$.
If column 1 of \underline{M}_1^* is 1-good then there is an $i_1'' > 2$ such that $R''(i_1'')$ intersects column 1 securely. We can move over to column 1 and escape to the cell below $R''(i_1'')$, charging its costs (similar to the above ones) to the scapegoat run of $R(1) \cap C(1)$, $R''(2)$.
- c. Suppose finally that no column of \underline{M}_1^* is 1-good, but that there is a j such that $|r_{j+1} - r_j| > 2$. We can then escape similarly to how we did in the big jump. The geometric cost is at most $B(40 + 10Q) \leq 11QB$, the number of moves is at most $10Q$. Charging is again not a problem just as there.

4.6. Big escape or big finish. A big escape is implemented just like a big continuing attack; the only difference is that we always have the straight case since the reduced body is safe.

A big finish is applied under some circumstances after a failed big attack with reduced body \underline{M}_0^* . The last move of the implementation of the big attack was either a failed attack, or a rightward step ending in a northwards bad position in the rightmost column of \underline{M}_0^* . If it was a failed attack we apply a small southward finish to arrive at the bottom cell of the failed attack or one cell below it, to be in a southward-clear point. In both cases then we step down to the cell above the second highest 1-clean row of the lowest colony of the big finish move. (The end result must be southward-clear.)

There is no extra geometric cost. The number of moves is at most $6Q$.

4.7. Big turn. A northward-eastward turn consists of a northward step with body M_0^* followed by an eastward jump or northward-sweeping eastward attack with body M_1^* . Of course, the destination colony D_0^* of the first part coincides with the start colony S_1^* of the second part.

Since the second part of the turn is a big eastward jump, in its discussion what were columns of the discussion of a big northward jump become rows and vice versa. Initially we are below row $R'(0)$ of S_0^* and in column $C(0)$ of M_0^* .

The key to the implementation of the turn is that there are κ clean columns of the big step, so we can direct the implementation of the big step in such a way as to arrive into an appropriate column of \underline{M}_1^* .

1. Suppose that we have the marginal case of M_1^* . Then we implement the big step in such a way that we arrive into the M_1^* along a clean column that crosses the first row of M_1^* securely. Then we turn east and after walking right near the obstacles begin a

northward sweep of a series of eastward attacks as in the implementation of an ordinary big eastward jump.

2. Suppose that we have the straight case of M_1^* . We then can direct the implementation of the big step M_0^* in each subcase in such a way that we will escape similarly to the corresponding subcase of the straight case of a big jump. For example if M_1^* has a 1-good row $R(1)$ then we will arrive along a 1-good column of M_0^* that intersects $R(1)$ securely.

We charge the extra geometric cost of the step to the geometric cost of the turn which is defined as $8QB$ instead of $5QB$ to accomodate this. The extra geometric cost of the jump can be charged as before.

This concludes the construction. It is easy to check that it satisfies the requirements of an implementation and thus the proves of the main lemma.

5. NESTED STRATEGIES

With the proof of the main lemma, it will be clear to some readers that the angel has a winning strategy. What follows is the formal definition of this strategy based on the implementation map. Let us first define the notions of strategy used.

Definition 5.1. Let Configs^+ be the set of nonempty finite sequences of configurations. We will use the addition notation

$$\gamma' + \mathbf{d} = \gamma$$

to add a new configuration to a sequence γ' . A *plain strategy* is a map

$$\eta : \text{Configs}^+ \rightarrow \text{Locmoves}$$

giving the angel's next move after each d-history.

We will call a plain strategy *winning* if it has the following property. For $n > 1$, let $(\mathbf{d}_1, \mathbf{a}_1, \dots, \mathbf{d}_n, \mathbf{a}_n)$ be an a-history in which

- (a) \mathbf{d}_1 is the default configuration, and for each $i > 1$, \mathbf{d}_i is a permitted move of the devil after the a-history $(\mathbf{d}_1, \dots, \mathbf{a}_{i-1})$.
- (b) For each $i \leq n$, we have $\mathbf{a}_i = \eta(\mathbf{d}_1, \dots, \mathbf{d}_i)$.

Then \mathbf{a}_n is a permitted move of the angel after the d-history $(\mathbf{d}_1, \mathbf{a}_1, \dots, \mathbf{d}_n)$. ┘

We will define a winning plain strategy with the help of nested strategies, which will be constructed with the help of the implementation map. We can scale the map of the lemma into maps (ϕ_k, J_k) and then use it to obtain an an amplifier for Φ_1, Φ_2, \dots where $\mathbb{G}_{k+1} = \mathbb{G}_k^*$, and \mathbb{G}_k has colony size $B_k = Q^k$.

Definition 5.2. A *stack* for game \mathbb{G}_k is a finite nonempty sequence of nonempty legal a-histories (χ_1, \dots, χ_m) , where $\chi_i \in \text{Histories}_a(B_{i+k-1})$, and if $m > 1$ then the last history χ_m is not the default record. It is understood that this finite sequence of a-histories stands for the infinite sequence in which each χ_i with $i > m$ is the default a-history (α_0) . Let

$$\text{Stacks}_k$$

be the set of all possible stacks for game \mathbb{G}_k . The interpretation of a stack can be that χ_i for $i \geq k$ is the history of game \mathbb{G}_i played so far, in a translation of the last step of game \mathbb{G}_{i+1} . (This interpretation imposes more restrictions on the possible stacks, but we do not need to spell these out.) ┘

Definition 5.3. A *nested strategy* for game \mathbb{G}_k is a map

$$\begin{aligned} \psi : \text{Configs} \times \text{Stacks}_k &\rightarrow \text{Stacks}_k, \\ (\mathbf{d}, (\chi_1, \dots, \chi_m)) &\mapsto \psi(\mathbf{d} \mid \chi_1, \dots, \chi_m). \end{aligned}$$

Here $\psi(\mathbf{d} \mid \chi_1, \dots, \chi_m) = (\chi'_1, \dots, \chi'_n)$ with $n \in \{m, m+1\}$. \lrcorner

The interpretation of a nested strategy for game \mathbb{G}_1 is the following. Consider an amplifier $\mathbb{G}_1, \mathbb{G}_2, \dots$. Then the angel of game \mathbb{G}_1 uses the the current configuration, further the following histories: the history χ_1 of game \mathbb{G}_1 since the beginning of the last step of game \mathbb{G}_2 (the earlier history of game \mathbb{G}_2 is not needed), the history χ_2 of game \mathbb{G}_2 since the beginning of the last step of game \mathbb{G}_3 , and so on.² The strategy computes the new next step of the angel and the corresponding new stack of histories. The following definition describes how this is done in our case.

Definition 5.4. Assume that we are given an amplifier $\mathbb{G}_1, \mathbb{G}_2, \dots$ with implementation maps $(\phi_1, J_1), (\phi_2, J_2), \dots$. We define a nested strategy ψ_k for each game \mathbb{G}_k (the functions ψ_k for different k actually differ only in the scale B_k). We want to compute

$$\Psi = \psi_k(\mathbf{d} \mid \chi_1, \dots, \chi_m).$$

The definition is by induction on the length m of the stack. If $m = 1$ then let $\chi_2 = (\alpha_0)$, the default a-history. Let $\alpha(\chi_2)$ be the last record of χ_2 , and

$$\mathbf{a} = \hat{\phi}_k(\chi_1 + \mathbf{d} \mid \alpha(\chi_2)). \quad (13)$$

If \mathbf{a} is not the halting move then $\Psi = (\chi_1 + \mathbf{d} + \mathbf{a}, \chi_2, \dots, \chi_m)$. Assume we have the halting move. Then we set

$$\begin{aligned} \mathbf{d}^* &= \hat{J}_k(\mathbf{d} \mid \chi_1, \dots, \chi_m), \\ (\chi'_2, \dots, \chi'_n) &= \psi_{k+1}(\mathbf{d}^* \mid \chi_2, \dots, \chi_m), \end{aligned} \quad (14)$$

$$\mathbf{a}' = \hat{\phi}_k((\mathbf{d}) \mid \alpha(\chi'_2)), \quad (15)$$

$$\Psi = ((\mathbf{d}, \mathbf{a}'), \chi'_2, \dots, \chi'_n).$$

In case $m = 1$ the step (14) does not lead to another recursive step. Indeed, then $\chi_2 = (\alpha_0)$ and then step (13) gives $\hat{\phi}_{k+1}(\alpha_0 + \mathbf{d}^* \mid \alpha(\chi_2))$. According to condition (b) of Definition 2.22, the result here cannot be the halting move. Step (15) cannot yield the halting move either. \lrcorner

It is easy to check by induction that the output of ψ_k is indeed a stack satisfying the requirements of Definition 5.2.

Let us derive a winning plain strategy from a nested strategy.

Definition 5.5. Let ψ be a nested strategy for game \mathbb{G}_1 . We define a plain strategy $\eta(\gamma)$ for \mathbb{G}_1 . As mentioned in the definition of plain strategies we only consider response histories γ which start with the default configuration \mathbf{d}_0 . We will make use of an auxiliary function

$$\hat{\eta} : \text{Configs}^+ \rightarrow \text{Stacks}_k.$$

Then if $\hat{\eta}(\gamma) = (\chi_1, \dots, \chi_m)$ we define $\eta(\gamma)$ as the last move of χ_1 .

²The reader may be amused by a faintly analogous idea in the poem *Ajedrez (Chess) II* by Jorge Louis Borges (findable on the internet). Borges refers back to Omar Khayyam.

The definition of $\hat{\eta}$ is by induction. For a sequence consisting of a single configuration we define

$$\hat{\eta}((\mathbf{d}_0)) = \psi(\mathbf{d}_0 \mid (\alpha_0)).$$

If $\gamma = \gamma' + \mathbf{d}$ then let

$$\begin{aligned} (\chi_1, \dots, \chi_m) &= \hat{\eta}(\gamma'), \\ \hat{\eta}(\gamma) &= \psi(\mathbf{d} \mid \chi_1, \dots, \chi_m). \end{aligned}$$

┘

The theorem below implies Theorem 1.

Theorem 2. *If (ϕ_k, J_k) is an implementation map for each k then the plain strategy η defined above is a winning strategy for the angel.*

Proof. The plain strategy of η was defined above via the nested strategy ψ . Tracing back the definition of ψ we see that the next move \mathbf{a}_i of the angel is always computed applying $\phi_1(\mathbf{d}_i \mid \alpha)$ for an appropriate record: see (13) and (15). Since ϕ_1 is an implementation map, the resulting move is always allowed. \square

6. CONCLUSIONS

One would think that a strategy depending only on the present position and measure should also be possible.

7. ACKNOWLEDGEMENT

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