Extra problems for induction

1. Prove by induction:
   Prove that, for all \( n \in \mathbb{N} \) where \( n \geq 1 \),
   \( 7^n - 1 \) is evenly divisible by 6.
2. Prove, for all \( n > 1 \),
   \( 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{(n^2(n + 1)^2)}{4} \)
3. Prove that \( 3^n > n^2 \) for \( n = 1, n = 2 \) and use mathematical induction to
   prove that \( 3^n > n^2 \) for \( n \) any positive integer greater than 2.

Extra problems for recurrence relations

1. Define the sequence \( x_1, x_2, x_3, \ldots \) by recursion, where \( x_1 = 3 \) and for all
   \( i > 1, x_{i+1} = 3x_i + 5 \).
   i. Write the first 5 \( x \) values.
   ii. What if the value for \( x_k \) in terms of \( k \)?
2. Let \( T(1) = 4 \) and \( T(n) = 2T(n-1) + 4 \)
   Write the first 5 terms of \( T \). See if you can guess what the nth term \( T(n) \)
   equals.
   And then try to prove it.
3. Solve the recurrence relation
   \( T(n) = 4T(n-1) - 3T(n-2), \) where \( T(0) = 0 \) and \( T(1) = 2 \).
   Prove by induction that
   \( T(n) = 3^n - 1 \).

ANSWERS:
Problem 1: Base case \( n+1 \). The \( 7^n - 1 = 7-1=6 \) is evenly divisible by 1.
Induction: Given \( 7^n - 1 \) evenly div. by 6, prove the same for \( 7^{n+1} - 1 \).
Well... \( 7^{n+1} - 1 = 7(7^n) - 1 = 6(7^n) + 7^n - 1 \)
Since \( 7^n - 1 \) evenly div. by 6 (this is the induction hypothesis), we have
\( 7^n - 1 = 6k \) for some integer \( k \).
Putting this together gives, \( 7^{n+1} - 1 = 6(7^n) + 7^n - 1 = 6(7^n) + 6k \) which is evenly divisible by 6 as we wanted to prove.
Problem 1, second part on recurrences.

By recursion, where \( x_1 = 3 \) and for all \( i > 1 \), \( x_{i+1} = 3x_i + 5 \).

\[
x_1 = 3, \quad x_2 = 3x_1 + 5 = 3(3) + 5 = 14, \quad x_3 = 3(14) + 5 = 42 + 5 = 47, \quad x_4 = 3x_3 + 5 = 3(47) + 5 = 146, \quad x_5 = 3x_4 + 5 = 3(146) + 5 = 443.
\]

So the pattern is 3, 14, 47, 146, 443, ..., and (admittedly) is not that clear. But if you look at how it came about (the computation above) you see that \( x_n = 3^n + 3^{n-2}5 + 3^{n-3}5 + ... + 3^{n-n}5 \), for any \( n \geq 1 \).

Problem 3 - 2nd part on recurrence.

\( T(n) = 4T(n-1) - 3T(n-2) \), where \( T(0) = 0 \) and \( T(1) = 2 \)

Proof:
Base cases; \( n=0 \), \( T(0) = 0 = 3^0 - 1 \); \( n=1 \), \( T(1) = 2 = 3^1 - 1 \).
Induction step: Assume \( n \geq 2 \) and \( T(n) = 3^n - 1 \) and prove \( T(n+1) = 3^{n+1} - 1 \)

Well, ... \( T(n+1) = 4T((n+1)-1) - 3T((n+1)-2) = 4T(n) - 3T(n-1) \) which by the induction hypothesis is

\[
= 4(3^n - 1) - 3(3^{n-1} - 1) = 4(3^n) - 4 - (3^n - 3) = 3^{n+1} - 4 + 3 = 3^{n+1} - 1
\]
as was to be proved.