Extra problems for induction

1. Prove by induction:
   Prove that, for all \( n \in \mathbb{N} \) where \( n \geq 1 \),
   \( 7^n - 1 \) is evenly divisible by 6.
2. Prove, for all \( n > 1 \),
   \( 1^3 + 2^3 + 3^3 + \ldots + n^3 = (n^2(n + 1)^2)/4 \)
3. Prove that \( 3^n > n^2 \) for \( n = 1, n = 2 \) and use mathematical induction to
   prove that \( 3^n > n^2 \) for \( n \) any positive integer greater than 2.

Extra problems for recurrence relations

1. Define the sequence \( x_1, x_2, x_3, \ldots \) by recursion, where \( x_1 = 3 \) and for all
   \( i > 1, x_{i+1} = 3x_i + 5 \).
   i. Write the first 5 \( x \) values.
   ii. What if the value for \( x_k \) in terms of \( k \)?
2. Let \( T(1) = 4 \) and \( T(n) = 2T(n-1) + 4 \)
   Write the first 5 terms of \( T \). See if you can guess what the nth term \( T(n) \)
   equals.
   And then try to prove it.
3. Solve the recurrence relation
   \( T(n) = 4T(n-1) - 3T(n-2) \), where \( T(0) = 0 \) and \( T(1) = 2 \).
   Prove by induction that
   \( T(n) = 3^n - 1 \).

ANSWERS:
Problem 1: Base case \( n+1 \). The \( 7^n - 1 = 7-1 = 6 \) is evenly divisible by 1.
Induction: Given \( 7^n - 1 \) evenly div. by 6, prove the same for \( 7^{n+1} - 1 \).
Well... \( 7^{n+1} - 1 = 7(7^n) - 1 = 6(7^n) + 7^n - 1 \)
Since \( 7^n - 1 \) evenly div. by 6 (this is the induction hypothesis), we have
\( 7^n - 1 = 6k \) for some integer \( k \).
Putting this together gives, \( 7^{n+1} - 1 = 6(7^n) + 7^n - 1 = 6(7^n) + 6k \) which is
evenly divisible by 6 as we wanted to prove.
Problem 3 - 2nd part on recurrence.

T(n) = 4T(n-1) - 3T(n-2), where T(0) = 0 and T(1) = 2

Proof:

Base cases; n=0, T(0) = 0 = 3^0 - 1; n=1, T(1) = 2 = 3^1 - 1.

Induction step: Assume n ≥ 2 and T(n) = 3^n - 1 and prove T(n+1)

= 3^{n+1} - 1

Well, ... T(n+1) = 4T((n+1)-1) - 3T((n+1)-2) = 4T(n) - 3T(n-1) = 4(3^n - 1) - 3(3^{n-1} - 1) = 4(3^n - 1) - (3^{n-1} - 3) = 3^n - 4 + 3 = 3^n - 1 as was to be proved.