Exercise 1 (20 points) Show that we can represent pairs of natural numbers using only natural numbers and arithmetic operations if we represent the pair \(a\) and \(b\) as the natural number \(2^a 3^b\). Please give the corresponding definitions of the procedures cons, car and cdr.

Exercise 2 (25 points) Let us represent a poker card as a pair \((\text{cons } s\ v)\), where \(1 \leq s \leq 4\) represents the suit of the card and \(1 \leq v \leq 13\) represents the value of the card. Please implement a procedure that returns a randomly shuffled deck of 52 distinct cards.

Exercise 3 (15 points) Please define the following function in a continuation-passing style.

\[
\text{(define (map f xs)}
\quad \text{(if (null? xs) ’() (cons (f (car x)) (map f (cdr xs)))))}
\]

Exercise 4 (20 points) Let \text{deep-reverse} be the function that takes a given list as its argument and returns as a value the list with the elements reversed and each element deep-reversed if the element is also a list. For instance, reversing the list \(((1\ 2)\ (3\ 4))\) yields \(((3\ 4)\ (1\ 2))\), while deep-reversing it gives \(((4\ 3)\ (2\ 1))\).

Exercise 5 (20 points) What would the interpreter print in response to evaluating each of the following expressions?

\[
\text{(list ’a ’b ’(c d))}
\]

\[
\text{(list (list ’Abraham))}
\]

\[
\text{(list ’(list Abraham))}
\]

\[
\text{(list ’(list ’Abraham))}
\]

\[
\text{(cdr ’((x1 y1) (x2 y2)))}
\]

\[
\text{(cadr ’((x1 y1) (x2 y2)))}
\]

\[
\text{(pair? (cddr ’(a beautiful rose)))}
\]
Exercise 6 (60 points) Suppose that we represent a vector \( v = (v_i) \) as a sequence of numbers, and a matrix \( A = (a_{ij}) \) as sequences of vectors (the rows of the matrix). For example, the matrix

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 & 11 & 12
\end{bmatrix}
\]

is represented as the following sequence.

\[
((1 2 3) (4 5 6) (7 8 9) (10 11 12))
\]

For a matrix \( A \) of dimension \( m \times n \), we use \((\text{cons } m (\text{cons } n A))\) for its representation, where \( m \), \( n \) and \( A \) are the representations of \( m \), \( n \) and \( A \), respectively.

1. (10 points) Implement a procedure that checks whether its argument is a correct representation of a matrix with dimension information.

2. (20 points) Implement a procedure that transposes a matrix with dimension information. For instance, given the argument below,

\[
(4 3 (1 2 3) (4 5 6) (7 8 9) (10 11 12))
\]

the procedure should return the following.

\[
(3 4 (1 4 7 10) (2 5 8 11) (3 6 9 12))
\]

In particular, if you transpose a matrix twice, you obtain the original matrix.

3. (30 points) Implement a procedure that takes two matrices and returns their product. Note that two matrices can be multiplied only if they are of dimensions \( p \times q \) and \( q \times r \) for some natural numbers \( p, q, r \).