Solution Keys to Assignment 1
BU CAS CS520: Principles of Programming Languages, Fall 2002

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[Exercise 1]
We prove it by induction on $n$.

**Base case:** If $n = 0$, then $(2n + 1)^2 - 1 = 0$, which is obvious a multiple of 8.

**Induction case:** If $n = k + 1$, we have the induction hypothesis that, $(2k + 1)^2 - 1$ is a multiple of 8, and our goal is to prove that $(2(k + 1) + 1)^2 - 1$ is also multiple of 8.

We know that:

$$(2(k + 1) + 1)^2 - 1 = ((2k + 1)^2 + 2) - 1 = ((2k + 1)^2 - 1) + 4(2k + 1) + 4 = ((2k + 1)^2 - 1) + 8(k + 1)$$

By induction hypothesis, we know that $(2k + 1)^2 - 1$ is multiple of 8, and we can see that $8(k + 1)$ is a multiple of 8. So we have $(2(k + 1) + 1)^2 - 1$ is also multiple of 8.

By these two cases, we complete the proof.

[Exercise 2]

1. Prove by structural induction on the formation of Braun Trees.

**Base case:** If $t = B(E, E)$, then $B(t) = 1$, and $h(t) = 1$. It is obvious that $2^{h(t) - 1} \leq B(t) \leq 2^{h(t)} - 1$.

**Induction case:** If $t = B(t_l, t_r)$, we have the induction hypothesis that, $2^{h(t_l) - 1} \leq B(t_l) \leq 2^{h(t_l)} - 1$, and $2^{h(t_r) - 1} \leq B(t_r) \leq 2^{h(t_r)} - 1$. Our goal is to prove $2^{h(t) - 1} \leq B(t) \leq 2^{h(t)} - 1$.

On one hand, we know that:

$$B(t) = B(t_l) + B(t_r) + 1$$

$$\geq B(t_l) + B(t_l) \quad \text{(by properties of Braun trees)}$$

$$\geq 2 \cdot 2^{h(t_l) - 1} \quad \text{(by induction hypothesis)}$$

$$= 2^{h(t_l)}$$
We also know that:

\[
B(t) = B(t_l) + B(t_r) + 1
\]
\[
\geq B(t_r) + B(t_r) + 1 \quad \text{(by properties of Braun trees)}
\]
\[
\geq 2 \cdot 2^{h(t_r) - 1} \quad \text{(by induction hypothesis)}
\]
\[
= 2^{h(t_r)}
\]

By these two in-equations, we know that:

\[
B(t) \geq \max(2^{h(t_l)}, 2^{h(t_r)})
\]
\[
= 2^{\max(h(t_l), h(t_r))}
\]
\[
= 2^{h(t) - 1} \quad \text{(by the definition of tree height)}
\]

On the other hand, we have:

\[
B(t) = B(t_l) + B(t_r) + 1
\]
\[
\leq (2^{h(t_l)} - 1) + (2^{h(t_r)} - 1) + 1
\]
\[
\leq 2 \cdot 2^{\max(h(t_l), h(t_r))} - 1
\]
\[
= 2^{h(t)} - 1 \quad \text{(by the definition of tree height)}
\]

By these two cases, we complete the proof.

3. We claim that for any \(n \geq 1\), there are \(2^{n-1}\) different Braun trees of height \(n\), and prove it by induction on \(n\).

**Base case:** If \(n = 1\), then the only possibility is that \(t = B(E, E)\). So the our claim is right for this case.

**Induction case:** If \(n = k + 1\), then we have the induction hypothesis that, there are \(2^{k-1}\) different Braun trees of height \(k\), and our goal is to prove that there are \(2^{k}\) different Braun trees of height \(k + 1\).

In order to prove this goal, we first prove the following lemma:

**Lemma 1** Given a number \(n\), there is exactly one Braun tree \(t\) such that \(B(t) = n\).

We prove Lemma 1 by induction on \(n\).

**Base case:** If \(n = 1\), then it must be \(t = B(E, E)\). So Lemma 1 holds for this case.

**Induction case:** If \(n = k + 1\), then we have the induction hypothesis that for any \(j \leq k\), there exists exactly one Braun tree of size \(j\), and our goal is to prove that there exists exactly on Braun tree of size \(k + 1\). This goal can be proved by the following case study:

**Case 1:** If \(k\) is an odd number, then we can assume \(k = 2 \cdot m - 1\). Then for any Braun tree \(t = B(t_l, t_r)\) of size \(k + 1\), it must be the case that \(B(t_l) = m \leq k\), and \(B(t_r) = m - 1 \leq k\). By induction hypothesis, we know that both \(t_l\) and \(t_r\) are unique. So \(t\) must be unique.

**Case 2:** If \(k\) is an even number, then we can assume \(k = 2 \cdot m\). Then for any Braun tree \(t = B(t_l, t_r)\) of size \(k + 1\), it must be the case that \(B(t_l) = B(t_r) = m \leq k\). By induction hypothesis, we know that both \(t_l\) and \(t_r\) are unique. So \(t\) must be unique.
So Lemma 1 also holds for the induction case.

By these cases, we complete the proof of Lemma 1.

Now we go back to our main proof. A direct corollary of Lemma 1 is that, for any Braun tree \( t \), \( h(t) = 1 + h(t_l) \). So for a Braun tree \( t = B(t_l, t_r) \) of height \( k + 1 \), we know \( h(t_l) = k \). By induction hypothesis, there are \( 2^{k-1} \) possible formations for \( t_l \). And for a certain formation of \( t_l \), there can be two possible cases for \( t_r \): either \( B(t_r) = B(t_l) \), or \( B(t_r) = B(t_l) - 1 \). By Lemma 1, we know for both cases, there is exactly one possible formation of \( t_r \). So there can be \( 2^{k-1} \cdot 2 = 2^k \) possible cases for \( t \). So our claim is also right for the induction case.

By these cases, we complete the proof for our claim.

\[\text{Exercise 3}\]

We prove it by the following case study:

**Case 1**: \( 92 \leq n \leq 101 \). We prove this case by induction on \( n \).

**Base case**: \( f_{91}(101) = f_{91}(101-10) = f_{91}(91) = 91 \).

**Induction case**: \( n = k - 1 \), where \( 93 \leq k \leq 101 \). We have the induction hypothesis that, \( f_{91}(k) = 91 \). Our goal is to prove that \( f_{91}(k-1) = 91 \).

We know that
\[
\begin{align*}
  f_{91}(k-1) &= f_{91}(f_{91}(k-1+11)) \quad (91 < k - 1 \leq 100) \\
  &= f_{91}(f_{91}(k-1+11-10)) \quad (102 < k - 1 + 11) \\
  &= f_{91}(f_{91}(k)) \\
  &= f_{91}(91) \quad \text{(by induction hypothesis)} \\
  &= 91 \quad \text{(by the definition of } f_{91})
\end{align*}
\]

By these cases, **Case 1** is proved.

**Case 2**: \( n \geq 102 \). Again, we prove this case by induction on \( n \).

**Base case**: \( f_{91}(102) = f_{91}(102-10) = f_{91}(92) = 91 \).

**Induction case**: \( n = k + 1 \), where \( k \geq 102 \). Combining the result of **Case 1** and the induction hypothesis, we have for any \( 92 \leq j \leq k \), \( f_{91}(j) = 91 \). Our goal is to prove that \( f_{91}(k+1) = 91 \).

We have
\[
\begin{align*}
  f_{91}(k+1) &= f_{91}(k+1-10) \quad (k + 1 \geq 103) \\
  &= 91 \quad (92 \leq k + 1 - 10 \leq k)
\end{align*}
\]

By these cases, **Case 2** is also proved.

**Case 3**: \( 0 \leq n \leq 90 \). The proof of this case is similar to that of **Case 2**.

By these three cases, we complete the proof.