Solution Keys to Assignment 1

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[Exercise 1]

We prove it by induction on n.

Base case: If $n = 0$, then $(2n+1)^2 - 1 = 0$, which is obvious a multiple of 8.

Induction case: If $n = k+1$, we have the induction hypothesis that, $(2k+1)^2 - 1$ is a multiple of 8, and our goal is to prove that $(2(k+1)+1)^2 - 1$ is also multiple of 8.

We know that:

$$\begin{array}{cl} & (2(k+1)+1)^2 - 1 \\ = & ((2k+1)+2)^2 - 1 \\ = & ((2k+1)^2 - 1) + 4(2k+1) + 4 \\ = & ((2k+1)^2 - 1) + 4(2k+2) \\ = & ((2k+1)^2 - 1) + 8(k+1) \end{array}$$

By induction hypothesis, we know that $(2k+1)^2 - 1$ is multiple of 8, and we can see that $8(k+1)$ is a multiple of 8. So we have $(2(k+1)+1)^2 - 1$ is also multiple of 8.

By these two cases, we complete the proof.

[Exercise 2]

1. Prove by structural induction on the formation of Braun Trees.

Base case: If $t = B(E, E)$, then $B(t) = 1$, and $h(t) = 1$. It is obvious that $2^{h(t)-1} \leq B(t) \leq 2^{h(t)} - 1$.

Induction case: If $t = B(t_l, t_r)$, we have the induction hypothesis that, $2^{h(t_l)-1} \leq B(t_l) \leq 2^{h(t_l)} - 1$, and $2^{h(t_r)-1} \leq B(t_r) \leq 2^{h(t_r)} - 1$. Our goal is to prove $2^{h(t)-1} \leq B(t) \leq 2^{h(t)} - 1$.

On one hand, we know that:

$$\begin{array}{rcll} B(t) & = & B(t_l) + B(t_r) + 1 & \\ & \geq & B(t_l) + B(t_l) & \text{(by properties of Braun trees)} \\ & \geq & 2 * 2^{h(t_l)-1} & \text{(by induction hypothesis)} \\ & = & 2^{h(t_l)} & \end{array}$$

We also know that:

$$\begin{array}{rcll} B(t) & = & B(t_l) + B(t_r) + 1 & \\ & \geq & B(t_r) + B(t_r) + 1 & \text{(by properties of Braun trees)} \\ & \geq & 2 * 2^{h(t_r)-1} & \text{(by induction hypothesis)} \\ & = & 2^{h(t_r)} & \end{array}$$

By these two in-equations, we know that:

$$\begin{array}{rcll} B(t) & \geq & max(2^{h(t_l)}, 2^{h(t_r)}) & \\ & = & 2^{max(h(t_l),h(t_r))} & \\ & = & 2^{h(t)-1} & \text{(by the definition of tree height)} \end{array}$$

On the other hand, we have:

$$\begin{array}{rcll} B(t) & = & B(t_l) + B(t_r) + 1 & \\ & \leq & (2^{h(t_l)} - 1) + (2^{h(t_r)} - 1) + 1 & \\ & \leq & 2 * 2^{max(h(t_l),h(t_r))} - 1 & \\ & = & 2^{h(t)} - 1 & \text{(by the definition of tree height)} \end{array}$$

By these two cases, we complete the proof.

3. We claim that for any $n \geq 1$, there are 2^{n-1} different Braun trees of height n, and prove it by induction on n.

Base case: If $n = 1$, then the only possibility is that $t = B(E, E)$. So the our claim is right for this case.

Induction case: If $n = k+1$, then we have the induction hypothesis that, there are 2^{k-1} different Braun trees of height k, and our goal is to prove that there are 2^k different Braun trees of height $k + 1$.

In order to prove this goal, we first prove the following lemma:

Lemma 1 *Given a number n, there is exactly one Braun tree t such that $B(t) = n$.*

We prove Lemma 1 by induction on n.

Base case: If $n = 1$, then it must be $t = B(E, E)$. So Lemma 1 holds for this case.

Induction case: If $n = k + 1$, then we have the induction hypothesis that for any $j \leq k$, there exists exactly one Braun tree of size j, and our goal is to prove that there exists exactly on Braun tree of size $k + 1$. This goal can be proved by the following case study:

Case 1: If k is an odd number, then we can assume $k = 2 * m - 1$. Then for any Braun tree $t = B(t_l, t_r)$ of size $k + 1$, it must be the case that $B(t_l) = m \leq k$, and $B(t_r) = m - 1 \leq k$. By induction hypothesis, we know that both t_l and t_r are unique. So t must be unique.

Case 2: If k is an even number, then we can assume $k = 2*m$. Then for any Braun tree $t = B(t_l, t_r)$ of size $k + 1$, it must be the case that $B(t_l) = B(t_r) = m \leq k$. By induction hypothesis, we know that both t_l and t_r are unique. So t must be unique.

So Lemma 1 also holds for the induction case.

By these cases, we complete the proof of the Lemma 1.

Now we go back to our main proof. A direct corollary of Lemma 1 is that, for any Braun tree t, $h(t) = 1 + h(t_l)$. So for a Braun tree $t = B(t_l, t_r)$ of height $k+1$, we know $h(t_l) = k$. By induction hypothesis, there are 2^{k-1} possible formations for t_l. And for a certain formation of t_l, there can be two possible cases for t_r: either $B(t_r) = B(t_l)$, or $B(t_r) = B(t_l) - 1$. By Lemma 1, we know for both cases, there is exactly one possible formation of t_r. So there can be $2^{k-1} * 2 = 2^k$ possible cases for t. So our claim is also right for the induction case.

By these cases, we complete the proof for our claim.

[Exercise 3]

We prove it by the following case study:

Case 1: $92 \le n \le 101$. We prove this case by induction on n.

Base case: $f91(101) = f91(101 - 10) = f(91) = 91$.

Induction case: $n = k - 1$, where $93 \le k \le 101$. We have the induction hypothesis that, $f91(k) = 91$. Our goal is to prove that $f91(k-1) = 91$.

We know that

$$
\begin{array}{rll}
 & f91(k-1) & \\
= & f91(f91(k-1+11)) & (91 < k-1 \le 100) \\
= & f91(f91(k-1+11-10)) & (102 < k-1+11) \\
= & f91(f91(k)) & \\
= & f91(91) & \text{(by induction hypothesis)} \\
= & 91 & \text{(by the definition of } f91\text{)}
\end{array}
$$

By these cases, **Case 1** is proved.

Case 2: $n \ge 102$. Again, we prove this case by induction on n.

Base case: $f91(102) = f91(102 - 10) = f91(92) = 91$.

Induction case: $n = k + 1$, where $k \ge 102$. Combining the result of **Case 1** and the induction hypothesis, we have for any $92 \le j \le k$, $f91(j) = 91$. Our goal is to prove that $f91(k+1) = 91$. We have

$$
\begin{array}{rll}
 & f91(k+1) & \\
= & f91(k+1-10) & (k+1 \ge 103) \\
= & 91 & (92 \le k+1-10 \le k)
\end{array}
$$

By these cases, **Case 2** is also proved.

Case 3: $0 \le n \le 90$. The proof of this case is similar to that of **Case 2**.

By these three cases, we complete the proof.