[Question 1] Extend the proof technique from Chapter 12 to show that simply typed lambda-calculus remains normalizing when extended with booleans (Figure 3-1) and lists (Figure 11-13).

Definition 1 We extend the definition of the predicate $R_T$ as follows.

- $R_{Bool}(t)$ iff $t \downarrow$
- $R_{val_{List[T]}}(v)$ iff
  - $v = nil[T]$, or
  - $v = \text{cons}[T](v_1)(v_2)$ and $R_T(v_1)$ and $R_{List[T]_1}(v_2)$.

Note that $R_{val_{List[T]}}(v)$ is defined by structural induction $v$.

- $R_{List[T]}(t)$ iff $t \downarrow$ and whenever $t \rightarrow^* v$, $R_{val_{List[T]}}(v)$.

Proposition 1 We prove that for each type $T$, $R_T$ has the following properties.

1. If $R_T(t)$ holds, then $t \downarrow$.
2. If $R_T(t)$ holds and $t \rightarrow t'$, then $R_T(t')$ holds.
3. If $t$ is not a value and $R_T(t')$ holds whenever $t \rightarrow t'$, then $R_T(t)$ holds.

Proof We just need to show the cases for $Bool$ and $List[T]$. Case 1: $T = Bool$. Then the definition of $R_T$ clearly implies that $R_T$ has the above properties.

Case 2: $T = List[T_1]$. Then we have the following.

1. Clearly, $R_T(t)$ implies $t \downarrow$.
2. Assume $R_T(t)$ holds and $t \rightarrow t'$. For each $v$ such that $t' \rightarrow^* v$, we have $t \rightarrow^* v$ and therefore $R_{val_{List[T]}}(v)$. Hence, $R_T(t')$ holds.
3. Assume $t$ is not a value and $R_T(t')$ holds for each $t'$ satisfying $t \rightarrow t'$. For each $v$ such that $t \rightarrow^* v$, we have $t \rightarrow t' \rightarrow^* v$ for some $t'$ since $t$ is not a value, and therefore $R_{val_{List[T]}}(v)$ holds. By the definition of $R_T$, we have $R_T(t)$.
Corollary 1 If $t$ is not a value and $R_T(v)$ for each $v$ satisfying $t \rightarrow^* v$, then $R_T(t)$.

**Proof** This immediately follows from Proposition 1 (3).

Lemma 1 Finally we extend the proof of the main lemma as follows. If $\Gamma \vdash t : T$, and $\vdash \theta : \Gamma$, and for each $x \in \text{dom}(\theta) = \text{dom}(\Gamma)$, $R_{\Gamma(x)}(\theta(x))$ holds, then $R_\tau(t[\theta])$ holds.

**Proof** By induction on the typing derivation $\mathcal{D}$ of $\Gamma \vdash t : T$.

Case 1: $\mathcal{D}$ is of the following form.

| $\Gamma \vdash c : \text{Bool}$ |

Then the case is trivial.

Case 2: $\mathcal{D}$ is of the following form,

| $\mathcal{D}_1 :: \Gamma \vdash t_1 : \text{List}[T_1]$ |
| $\Gamma \vdash \text{isNil}[T_1](t_1) : \text{Bool}$ |

where $T = \text{Bool}$ and $t = \text{isNil}[T_1](t_1)$. Then by induction hypothesis on $\mathcal{D}_1$, we know $R_{\text{List}[T_1]}(t_1[\theta])$ holds. Hence we have $t_1[\theta] \downarrow$, which implies $t[\theta] \downarrow$. Thus, $R_{\text{Bool}}(t[\theta])$ holds by the definition of $R_{\text{Bool}}$.

Case 3: $\mathcal{D}$ is of the following form,

| $\Gamma \vdash \text{nil}[T_1] : \text{List}[T_1]$ |

where $T = \text{List}[T_1]$. This case is trivial.

Case 4: $\mathcal{D}$ is of the following form,

| $\mathcal{D}_1 :: \Gamma \vdash t_1 : T_1$ |
| $\mathcal{D}_2 :: \Gamma \vdash t_2 : \text{List}[T_1]$ |

where $T = \text{List}[T_1]$ and $t = \text{cons}[T_1](t_1)(t_2)$. Then by induction hypotheses on $\mathcal{D}_1$ and $\mathcal{D}_2$, we have $R_T(t_1[\theta])$ and $R_{\text{List}[T_1]}(t_2[\theta])$. Hence we have $t_1[\theta] \downarrow$ and $t_2[\theta] \downarrow$, which imply $t[\theta] \downarrow$. Assume $t[\theta] = \text{cons}[T_1](t_1[\theta])(t_2[\theta]) \rightarrow^* v$. Then $v = \text{cons}(v_1)(v_2)$ and $t_1[\theta] \rightarrow^* v_1$ and $t_2[\theta] \rightarrow^* v_2$. Hence, we have $R_{T_1}(v_1)$ and $R_{\text{List}[T_1]}(v_2)$, which imply $R_{\text{List}[T_1]}(\text{cons}([T_1](v_1)(v_2)))$. By the definition of $R_{\text{List}[T_1]}$, we have $R_{\text{List}[T_1]}(t[\theta])$.

Case 5: $\mathcal{D}$ is of the following form,

| $\Gamma \vdash t_1 : \text{List}[T_1]$ |

where $T = T_1$ and $t = \text{hd}[T_1](t_1)$. Then by induction hypothesis, we have $R_{\text{List}[T_1]}(t_1[\theta])$. Hence, we have $t_1[\theta] \downarrow$, which implies $t[\theta] = \text{hd}[T_1](t_1[\theta]) \downarrow$. Assume $t[\theta] \rightarrow^* v_1$. Then $t_1[\theta] \rightarrow^* \text{cons}[T_1](v_1)(v_2)$ for some $v_2$ and $R_{\text{List}[T_1]}(\text{cons}([T_1](v_1)(v_2)))$ holds. Hence, $R_{T_1}(v_1)$ holds. By Corollary 1, we have $R_{T_1}(t[\theta])$.

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Case 6: $D$ is of the following form,

\[
\frac{
\Gamma \vdash t_1 : List[T_1]
}{
\Gamma \vdash \text{tl}[T_1](t_1) : List[T_1]
}\]

where $T = List[T_1]$ and $t = \text{tl}[T_1](t_1)$. Then by induction hypothesis, we have $R_{List[T_1]}(t_1[\theta])$.

Assume that $\text{tl}[T_1](t_1[\theta]) \rightarrow^* v_2$. Then $t_1[\theta] \rightarrow^* \text{cons}[T_1](v_1)(v_2)$ for some $v_1$ and $R_{List[T_1]}^{val}(v_2)$ holds. By the definition of $R_{List[T_1]}$, we have $R_{List[T_1]}(t[\theta])$. 