

Solution Keys to Assignment 4

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[Question 1] Extend the proof technique from Chapter 12 to show that simply typed lambda-calculus remains normalizing when extended with booleans (Figure 3-1) and lists (Figure 11-13).

Definition 1 We extend the definition of the predicate R_T as follows.

- $R_{Bool}(t)$ iff $t \downarrow$.
- $R_{List[T]}^{val}(v)$ iff
 - $v = nil[T]$, or
 - $v = cons[T](v_1)(v_2)$ and $R_T(v_1)$ and $R_{List[T]}(v_2)$.

Note that $R_{List[T]}^{val}(v)$ is defined by structural induction on v .

- $R^{List[T]}(t)$ iff $t \downarrow$ and whenever $t \rightarrow^* v$, $R_{List[T]}^{val}(v)$.

Proposition 1 We prove that for each type T , R_T has the following properties.

1. If $R_T(t)$ holds, then $t \downarrow$.
2. If $R_T(t)$ holds and $t \rightarrow t'$, then $R_T(t')$ holds.
3. If t is not a value and $R_T(t')$ holds whenever $t \rightarrow t'$, then $R_T(t)$ holds.

Proof We just need to show the cases for $Bool$ and $List[T]$. **Case 1:** $T = Bool$. Then the definition of R_T clearly implies that R_T has the above properties .

Case 2: $T = List[T_1]$. Then we have the following.

- (1). Clearly, $R_T(t)$ implies $t \downarrow$.
- (2). Assume $R_T(t)$ holds and $t \rightarrow t'$. For each v such that $t' \rightarrow^* v$, we have $t \rightarrow^* v$ and therefore $R_{List[T]}^{val}(v)$. Hence, $R_T(t')$ holds.
- (3). Assume t is not a value and $R_T(t')$ holds for each t' satisfying $t \rightarrow t'$. For each v such that $t \rightarrow^* v$, we have $t \rightarrow t' \rightarrow^* v$ for some t' since t is not a value, and therefore $R_{List[T]}^{val}(v)$ holds. By the definition of R_T , we have $R_T(t)$.

Corollary 1 *If t is not a value and $R_T(v)$ for each v satisfying $t \rightarrow^* v$, then $R_T(t)$.*

Proof This immediately follows from Proposition 1 (3).

Lemma 1 *Finally we extend the proof of the main lemma as follows. If $\Gamma \vdash t : T$, and $\vdash \theta : \Gamma$, and for each $x \in \mathbf{dom}(\theta) = \mathbf{dom}(\Gamma)$, $R_{\Gamma(x)}(\theta(x))$ holds, then $R_\tau(t[\theta])$ holds.*

Proof By induction on the typing derivation \mathcal{D} of $\Gamma \vdash t : T$.

Case 1: \mathcal{D} is of the following form.

$$\overline{\Gamma \vdash c : Bool}$$

Then the case is trivial.

Case 2: \mathcal{D} is of the following form,

$$\frac{\mathcal{D}_1 :: \Gamma \vdash t_1 : List[T_1]}{\Gamma \vdash \mathbf{isNil}[T_1](t_1) : Bool}$$

where $T = Bool$ and $t = \mathbf{isNil}[T_1](t_1)$. Then by induction hypothesis on \mathcal{D}_1 , we know $R_{List[T_1]}(t_1[\theta])$ holds. Hence we have $t_1[\theta] \downarrow$, which implies $t[\theta] \downarrow$. Thus, $R_{Bool}(t[\theta])$ holds by the definition of R_{Bool} .

Case 3: \mathcal{D} is of the following form,

$$\overline{\Gamma \vdash \mathbf{nil}[T_1] : List[T_1]}$$

where $T = List[T_1]$. This case is trivial.

Case 4: \mathcal{D} is of the following form,

$$\frac{\mathcal{D}_1 :: \Gamma \vdash t_1 : T_1 \quad \mathcal{D}_2 :: \Gamma \vdash t_2 : List[T_1]}{\Gamma \vdash \mathbf{cons}[T_1](t_1)(t_2) : List[T_1]}$$

where $T = List[T_1]$ and $t = \mathbf{cons}[T_1](t_1)(t_2)$. Then by induction hypotheses on \mathcal{D}_1 and \mathcal{D}_2 , we have $R_{T_1}(t_1[\theta])$ and $R_{List[T_1]}(t_2[\theta])$. Hence we have $t_1[\theta] \downarrow$ and $t_2[\theta] \downarrow$, which imply $t[\theta] \downarrow$. Assume $t[\theta] = \mathbf{cons}[T_1](t_1[\theta])(t_2[\theta]) \rightarrow^* v$. Then $v = \mathbf{cons}(v_1)(v_2)$ and $t_1[\theta] \rightarrow^* v_1$ and $t_2[\theta] \rightarrow^* v_2$. Hence, we have $R_{T_1}(v_1)$ and $R_{List[T_1]}^{val}(v_2)$, which imply $R_{List[T_1]}^{val}(\mathbf{cons}([T_1]v_1)(v_2))$. By the definition of $R_{List[T_1]}$, we have $R_{List[T_1]}(t[\theta])$.

Case 5: \mathcal{D} is of the following form,

$$\frac{\Gamma \vdash t_1 : List[T_1]}{\Gamma \vdash \mathbf{hd}[T_1](t_1) : T_1}$$

where $T = T_1$ and $t = \mathbf{hd}[T_1](t_1)$. Then by induction hypothesis, we have $R_{List[T_1]}(t_1[\theta])$. Hence, we have $t_1[\theta] \downarrow$, which implies $t[\theta] = \mathbf{hd}[T_1](t_1[\theta]) \downarrow$. Assume $t[\theta] \rightarrow^* v_1$. Then $t_1[\theta] \rightarrow^* \mathbf{cons}[T_1](v_1)(v_2)$ for some v_2 and $R^{val}(\mathbf{cons}[T_1](v_1)(v_2))$ holds. Hence, $R_{T_1}(v_1)$ holds. By Corollary 1, we have $R_{T_1}(t[\theta])$.

Case 6: \mathcal{D} is of the following form,

$$\frac{\Gamma \vdash t_1 : List[T_1]}{\Gamma \vdash \mathbf{tl}[T_1](t_1) : List[T_1]}$$

where $T = List[T_1]$ and $t = \mathbf{tl}[T_1](t_1)$. Then by induction hypothesis, we have $R_{List[T_1]}(t_1[\theta])$. Assume that $\mathbf{tl}[T_1](t_1[\theta]) \rightarrow^* v_2$. Then $t_1[\theta] \rightarrow^* \mathbf{cons}[T_1](v_1)(v_2)$ for some v_1 and $R_{List[T_1]}^{val}(v_2)$ holds. By the definition of $R_{List[T_1]}$, we have $R_{List[T_1]}(t[\theta])$.