BU CAS CS 520 (Fall Semester, 2003)
Principles of Programming Languages

Assignment 1

Out: Wednesday, 05 Septmeber 2003
Due: Monday, 15 September 2003

Academic Integrity We pledge strict adherence to the university guidelines.

- All work you turn in must be *your own* unless specified otherwise.
- You are allowed to discuss problems with your classmates but you must write your own code and solutions.
- Please always remeber that every student deserves a chance to achieve a fair grade!

Total: 180 points

Exercise 1 *(20 pts) Prove the following statement using mathematical induction.*

$$(2n+1)^2 - 1 \textit{ is a multiple of } 8 \textit{ for every natural number } n.$$

Exercise 2 *(120 pts) Binary trees are defined as follows.*

$$\textit{binary trees} \quad t \quad ::= \quad E \mid B(t,t)$$

Let us define the functions s and h on binary trees inductively:

$$\begin{array}{ll} s(E) = 0 & s(B(t_l, t_r)) = 1 + s(t_l) + s(t_r) \\ h(E) = 0 & h(B(t_l, t_r)) = 1 + \max(h(t_l), h(t_r)) \end{array}$$

Evidently, $s(t)$ and $h(t)$ compute the size and the height of t, respectively. Please prove the following proposition:

- *(30 pts) For every binary tree t, $s(t) \leq 2^{h(t)}$ holds.*

Braun trees are special binary trees defined as follows.

- *E is a Braun tree.*
- *$B(t_l, t_r)$ is a Braun tree if t_1 and t_r are Braun trees and $B(t_r) \leq B(t_l) \leq B(t_r) + 1$.*

Please do the following.

- *(30 pts) Prove by structural induction on t that for every nonempty Braun tree t, $2^{h(t)-1} \leq B(t) < 2^{h(t)}$ holds.*
- *(30 pts) Implement in SML a procedure that lists all Braun trees of height n when given a natural number n.*

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datatype BraunTree = E | B of BraunTree * BraunTree

fun listBraunTrees (height: int): BraunTree list = (* your code is here *)
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- *(30 pts) In general, what is the number of Braun trees of height n for a given natural number n? Please justify your answer by mathematical induction.*

Exercise 3 *(40 points) MacCarthy's 91-function is defined as follows.*

$$f91(n) = \begin{cases} 91 & \textit{if } n = 91; \\ f91(f91(n+11)) & \textit{if } 0 \leq n < 91 \textit{ or } 91 < n \leq 100; \\ f91(n-10) & \textit{if } n > 100; \end{cases}$$

Please use inductive reasoning to prove that $f91(n) = 91$ for every natural number n.