Assignment 1

Out: Wednesday, 05 September 2003
Due: Monday, 15 September 2003

Academic Integrity We pledge strict adherence to the university guidelines.

- All work you turn in must be your own unless specified otherwise.
- You are allowed to discuss problems with your classmates but you must write your own code and solutions.
- Please always remember that every student deserves a chance to achieve a fair grade!

Total: 180 points

Exercise 1 (20 pts) Prove the following statement using mathematical induction.

\[(2n + 1)^2 - 1\] is a multiple of 8 for every natural number \(n\).

Exercise 2 (120 pts) Binary trees are defined as follows.

\[
\text{binary trees } t := E \mid B(t, t)
\]

Let us define the functions \(s\) and \(h\) on binary trees inductively:

\[
\begin{align*}
    s(E) &= 0 \\
    h(E) &= 0 \\
    s(B(t_l, t_r)) &= 1 + s(t_l) + s(t_r) \\
    h(B(t_l, t_r)) &= 1 + \max(h(t_l), h(t_r))
\end{align*}
\]

Evidently, \(s(t)\) and \(h(t)\) compute the size and the height of \(t\), respectively. Please prove the following proposition:

- (30 pts) For every binary tree \(t\), \(s(t) \leq 2^{h(t)}\) holds.

Braun trees are special binary trees defined as follows.

- \(E\) is a Braun tree.
- \(B(t_l, t_r)\) is a Braun tree if \(t_l\) and \(t_r\) are Braun trees and \(B(t_r) \leq B(t_l) \leq B(t_r) + 1\).

Please do the following.

- (30 pts) Prove by structural induction on \(t\) that for every nonempty Braun tree \(t\), \(2^{h(t) - 1} \leq B(t) < 2^{h(t)}\) holds.
- (30 pts) Implement in SML a procedure that lists all Braun trees of height \(n\) when given a natural number \(n\).
datatype BraunTree = E | B of BraunTree * BraunTree

fun listBraunTrees (height: int): BraunTree list = (* your code is here *)

• (30 pts) In general, what is the number of Braun trees of height \( n \) for a given natural number \( n \)? Please justify your answer by mathematical induction.

Exercise 3 (40 points) MacCarthy’s 91-function is defined as follows.

\[
f_{91}(n) = \begin{cases} 
91 & \text{if } n = 91; \\
f_{91}(f_{91}(n + 11)) & \text{if } 0 \leq n < 91 \text{ or } 91 < n \leq 100; \\
91(n - 10) & \text{if } n > 100;
\end{cases}
\]

Please use inductive reasoning to prove that \( f_{91}(n) = 91 \) for every natural number \( n \).