Assignment 1

Out: Tuesday, 2 September 2008
Due: Tuesday, 16 September 2008

Academic Integrity We pledge strict adherence to the university guidelines.

- All work you turn in must be your own unless specified otherwise.
- You are allowed to discuss problems with your classmates but you must write your own code and solutions.
- Please always remember that every student deserves a chance to achieve a fair grade!

Total: 200 points

Exercise 1 (20 pts) Prove the following statement using mathematical induction.

\[(2n + 1)^2 - 1 \text{ is a multiple of 8 for every natural number } n.\]

Exercise 2 (90 pts) Binary trees are defined as follows and we use \(B(t)\) for the number of branch node \(B\) in \(t\).

\[
\text{binary trees } t \ := \ E \ | \ B(t,t)
\]

Let us define function \(h\) on binary trees inductively.

\[
h(E) = 0 \quad h(B(t_l, t_r)) = 1 + \max(h(t_l), h(t_r))
\]

Evidently, \(h(t)\) computes the height of \(t\). Braun trees are binary trees defined as follows.

- \(E\) is a Braun tree.
- \(B(t_l, t_r)\) is a Braun tree if \(t_l\) and \(t_r\) are Braun trees and \(B(t_l) \leq B(t) \leq B(t_r) + 1.\)

Please do the following.

- (30 pts) Prove by structural induction on \(t\) that for every nonempty Braun tree \(t\), \(2^{h(t)} - 1 \leq B(t) < 2^{h(t)}\) holds.

- (30 pts) Implement in ATS a procedure that lists all Braun trees of height \(n\) when given a natural number \(n\).

```
datatype BraunTree = E | B of (BraunTree, BraunTree)

fun listBraunTrees (height: int): list0 BraunTree = (* your code is here *)
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• (30 pts) In general, what is the number of Braun trees of height $n$ for a given natural number $n$? Please justify your answer by mathematical induction.

**Exercise 3 (40 points)** MacCarthy’s 91-function is defined as follows.

$$f_{91}(n) = \begin{cases} 
91 & \text{if } n = 91; \\
f_{91}(f_{91}(n + 11)) & \text{if } 0 \leq n < 91 \text{ or } 91 < n \leq 100; \\
f_{91}(n - 10) & \text{if } n > 100; 
\end{cases}$$

Please use inductive reasoning to prove that $f_{91}(n) = 91$ for every natural number $n$.

**Exercise 4 (50 points)** Please implement in ATS the Game-of-24 described as follows.

Given four natural numbers $n_1, n_2, n_3$ and $n_4$, one chooses two of them and generates a rational number $r_1$ using either addition, subtraction, multiplication or division; one mixes $r_1$ with the remaining two numbers and chooses two of them to generate a rational number $r_2$ using either addition, subtraction, multiplication or division; one then takes $r_2$ and the last remaining number to get a rational number $r_3$ using addition, subtraction, multiplication, or division; if there is a way to make $r_3$ equal to 24, then we say that $(n_1, n_2, n_3, n_4)$ is a good quad. For instance, $(10, 10, 4, 4)$ is a good quad since we have

$$(10 * 10 - 4)/4 = 24$$

Similarly, $(5, 7, 7, 11)$ is a good quad since we have

$$(5 - 11/7) * 7 = 24$$

Game of 24 is a game that determines whether four given natural numbers are a good quad. Please implement a program in ATS that takes four given natural numbers and returns 1 or 0 according to whether the four natural numbers are a good quad; if they are a good quad, the program should also print out an arithmetic expression that attests to their being a good quad.