BU CAS CS 520 (Fall Semester, 2008)
Principles of Programming Languages

Assignment 1

Out: Tuesday, 2 Septmeber 2008
Due: Tuesday, 16 September 2008

Academic Integrity We pledge strict adherence to the university guidelines.

- All work you turn in must be *your own* unless specified otherwise.
- You are allowed to discuss problems with your classmates but you must write your own code and solutions.
- Please always remeber that every student deserves a chance to achieve a fair grade!

Total: 200 points

Exercise 1 *(20 pts) Prove the following statement using mathematical induction.*

$$(2n+1)^2 - 1 \text{ is a multiple of } 8 \text{ for every natural number } n.$$

Exercise 2 *(90 pts) Binary trees are defined as follows and we use $B(t)$ for the number of branch node B in t.*

$$\textit{binary trees} \quad t \quad ::= \quad E \mid B(t,t)$$

Let us define function h on binary trees inductively.

$$h(E) = 0 \qquad\qquad h(B(t_l, t_r)) = 1 + \max(h(t_l), h(t_r))$$

Evidently, $h(t)$ computes the height of t. Braun trees are binary trees defined as follows.

- *E is a Braun tree.*
- *$B(t_l, t_r)$ is a Braun tree if t_1 and t_r are Braun trees and $B(t_r) \leq B(t_l) \leq B(t_r) + 1$.*

Please do the following.

- *(30 pts) Prove by structural induction on t that for every nonempty Braun tree t, $2^{h(t)-1} \leq B(t) < 2^{h(t)}$ holds.*
- *(30 pts) Implement in ATS a procedure that lists all Braun trees of height n when given a natural number n.*

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datatype BraunTree = E | B of (BraunTree, BraunTree)

fun listBraunTrees (height: int): list0 BraunTree = (* your code is here *)
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- *(30 pts) In general, what is the number of Braun trees of height n for a given natural number n? Please justify your answer by mathematical induction.*

Exercise 3 *(40 points) MacCarthy's 91-function is defined as follows.*

$$f91(n) = \begin{cases} 91 & \text{if } n = 91; \\ f91(f91(n+11)) & \text{if } 0 \leq n < 91 \text{ or } 91 < n \leq 100; \\ f91(n-10) & \text{if } n > 100; \end{cases}$$

Please use inductive reasoning to prove that $f91(n) = 91$ for every natural number n.

Exercise 4 *(50 points) Please implement in ATS the Game-of-24 described as follows.*

Given four natural numbers n_1, n_2, n_3 and n_4, one chooses two of them and generates a rational *number r_1 using either addition, subtraction, multiplication or division; one mixes r_1 with the remaining two numbers and chooses two of them to generate a rational number r_2 using either addition, subtraction, multiplication or division; one then takes r_2 and the last remaining number to get a rational number r_3 using addition, subtraction, multiplication, or division; if there is a way to make r_3 equal to 24, then we say that (n_1, n_2, n_3, n_4) is a good quad. For instance, $(10, 10, 4, 4)$ is a good quad since we have*

$$(10 * 10 - 4)/4 = 24$$

Similarly, $(5, 7, 7, 11)$ is a good quad since we have

$$(5 - 11/7) * 7 = 24$$

Game of 24 is a game that determines whether four given natural numbers are a good quad.

Please implement a program in ATS that takes four given natural numbers and returns 1 or 0 according to whether the four natural numbers are a good quad; if they are a good quad, the program should also print out an arithmetic expression that attests to their being a good quad.