BU CAS CS 520 (Fall Semester, 2002)
Principles of Programming Languages

Continuation

(Author: Hongwei Xi)

types	T	$::=$	$\ldots \mid \mathit{cont}(T_1, T_2)$
expressions	e	$::=$	$\ldots \mid \mathbf{callcc}(e) \mid \mathbf{throw}(e_1, e_2)$
eval. ctx.	E	$::=$	$\ldots \mid \mathbf{callcc}(E) \mid \mathbf{throw}(E, e)$
values	v	$::=$	$\ldots \mid E^*$

$$\frac{x : T_1 \vdash E[x] : T_2 \quad \Gamma \vdash e : \mathit{cont}(T_1, T_2) \to T_1}{\Gamma \vdash E[\mathbf{callcc}(e)] : T_2}\ (\textbf{ty-callcc})$$

$$\frac{x : T_0 \vdash E[x] : T_2 \quad \Gamma \vdash e_1 : \mathit{cont}(T_1, T_2) \quad \Gamma \vdash e_2 : T_1}{\Gamma \vdash E[\mathbf{throw}(e_1, e_2)] : T_2}\ (\textbf{ty-throw})$$

$$\frac{x : T_1 \vdash E[x] : T_2}{\Gamma \vdash E^* : \mathit{cont}(T_1, T_2)}\ (\textbf{ty-cont})$$

- $E[\mathbf{callcc}(\lambda x.e)]$ is a redex, and its reduct is $E[e[x := E^*]]$.
- $E[\mathbf{throw}(E_0^*, e)]$ is a redex, and its reduct is $E_0[e]$.

Lemma 1 *(Canonical Forms) Assume that $\emptyset \vdash v : T$ is derivable.*

- *If $T = \mathit{cont}(T_1, T_2)$, then $v = E^*$ for some evaluation context E.*

Theorem 2 *(Progress) Assume that $\emptyset \vdash e : T$ is derviable. Then either e is a value, or $e \to e'$ for some expression e'.*

Proof By structural induction on e. ■

Theorem 3 *(Subject Reduction) Assume that $\emptyset \vdash e : T$ is derviable and $e \to e'$. Then $\emptyset \vdash e' : T$ is derivable.*

Proof By structural induction on the derivation $\mathcal{D}$ of $\emptyset \vdash e : T$.

- The last applied rule in $\mathcal{D}$ is (**ty-callcc**). Then $\mathcal{D}$ is of the following form:

$$\frac{x : T_1 \vdash E[x] : T_2 \quad \emptyset \vdash e_0 : \mathit{cont}(T_1, T_2) \to T_1}{\emptyset \vdash E[\mathbf{callcc}(e_0)] : T_2}\ (\textbf{ty-callcc})$$

where $e = E[\textbf{callcc}(e_0)]$ and $T = T_2$.

If e_0 is not a value, then $e_0 \rightarrow e'_0$ for some e'_0 and $e' = \textbf{callcc}(e'_0)$. By induction hypothesis on $\mathcal{D}_1$, we know that $\emptyset \vdash e'_0 : \mathit{cont}(T_1, T_2)$ is derivable. Hence, $\emptyset \vdash e' : T_2$ can be derived as follows:

$$\frac{x : T_1 \vdash E[x] : T_2 \quad \mathcal{D}_1 :: \emptyset \vdash e'_0 : \mathit{cont}(T_1, T_2) \rightarrow T_1}{\emptyset \vdash E[\textbf{callcc}(e'_0)] : T_2} \ (\textbf{ty-callcc})$$

We now assume that e_0 is a value. By the lemma of canonical forms, e_0 must be of the form $\lambda x.e_1$. Hence, $e' = E[e_1[x := E^*]]$. Note that $\emptyset \vdash E^* : \mathit{cont}(T_1, T_2)$ is derivable. By the substitution lemma, we know that $\emptyset \vdash e_1[x := E^*] : T_1$ is derivable. Hence, $\emptyset \vdash e' : T_2$ is also derivable.

- The last applied rule in $\mathcal{D}$ is (**ty-throw**). Then $\mathcal{D}$ is of the following form:

$$\frac{x : T_0 \vdash E[x] : T_2 \quad \mathcal{D}_1 :: \Gamma \vdash e_1 : \mathit{cont}(T_1, T_2) \quad \mathcal{D}_2 :: \Gamma \vdash e_2 : T_1}{\Gamma \vdash E[\textbf{throw}(e_1, e_2)] : T_2} \ (\textbf{ty-throw})$$

where $e = E[\textbf{throw}(e_1, e_2)]$ and $T = T_2$.

If e_1 is not a value, then $e_1 \rightarrow e'_1$ holds for some e'_1 and $e' = \textbf{throw}(e'_1, e_2)$. By induction hypothesis on $\mathcal{D}_1$, $\emptyset \vdash e'_1 : \mathit{cont}(T_1, T_2)$ is derivable. Hence, we can derive $\emptyset \vdash e' : T$ as follows:

$$\frac{x : T_0 \vdash E[x] : T_2 \quad \mathcal{D}_1 :: \Gamma \vdash e'_1 : \mathit{cont}(T_1, T_2) \quad \mathcal{D}_2 :: \Gamma \vdash e_2 : T_1}{\Gamma \vdash E[\textbf{throw}(e'_1, e_2)] : T_2} \ (\textbf{ty-throw})$$

We now assume that e_1 is a value. By the lemma of canonical forms, e_1 must be of the form E^*_0 for some evaluation context E_0. Hence, $e' = E_0[e_2]$. Clearly, $x : T_1 \vdash E_0[x] : T_2$ is derivable, which implies that $\emptyset \vdash e' : T$ is also derivable.

The other cases have all been handled before. ■