



BU CAS CS 520 (FALL SEMESTER, 2009)
PRINCIPLES OF PROGRAMMING LANGUAGES

Assignment 4

Out: Monday, 19 October 2009
Due: Thursday, 29 October 2009

Total: 110 points + 30 extra points

Exercise 1 (20 points) Let us define $List(T)$ as $\forall X.X \rightarrow (T \rightarrow X \rightarrow X) \rightarrow X$. Then the two list constructors *Nil* and *Cons* can be defined as follows.

$$\begin{aligned} Nil &= \Lambda X.\lambda nil : X.\lambda cons : T \rightarrow X \rightarrow X.nil \\ Cons(x : T)(xs : List(T)) &= \Lambda X.\lambda nil : X.\lambda cons : T \rightarrow X \rightarrow X.cons(x)(xs[X](nil)(cons)) \end{aligned}$$

Given the above list encoding, please implement the *append* and *reverse* functions on lists in System F. Your implementation needs to be coded in ATS. Note that you may not construct recursive functions in your implementation.

Exercise 2 (20 points + 30 extra points) The following datatype *gtree* and

```
datatype gtree (a:t@ype) =
  | E (a) of () | B (a) of (a -<cloref1> gtree a)
```

The type constructor *gtree* can be used to form general tree types. For instance, *gtree*(*bool*) can be considered as a type for binary trees; $B(\lambda x \Rightarrow E)$ represents a binary tree t_1 whose left and right subtrees are empty; $B(\lambda x \Rightarrow \text{if } x \text{ then } E \text{ else } t_1)$ represents a binary tree t_2 whose left subtree is empty and right subtree is t_1 .

- (10 points) Please construct a value of the type *gtree*(*int*) in ATS such that this value represents an infinite tree satisfying the following property:
 - For each $n \geq 0$, every node at level n has $n + 1$ nonempty children.

For instance, the root is at level 0, and it has one nonempty child; this child is at level 1, and it has two nonempty children, which are at level 2; etc.

- (10 points) Please encode *gtree* and its associated constructors *E* and *B* in System F.
- (30 points) Please implement the following function *leftGtree* in System F. Your implementation needs to be written in ATS.

```
fun leftGtree (t: gtree bool) =
  case+ t of
  | B (ft) => let
    val ft = ft: bool -<cloref1> gtree bool in ft (true)
  end // end of [B]
  | E () => E ()
```

Note that you may not construct recursive functions in your implementation of `leftGtree`.

Exercise 3 (20 points) We can define a type erasure function $|\cdot|$ as follows, which translates a term in System F into an untyped λ -term.

$$|x| = x \quad |\lambda x : T.t| = \lambda x. |t| \quad |t_1(t_2)| = |t_1|(|t_2|) \quad |\Lambda X.t| = |t| \quad |t[T]| = |t|$$

Notice that for a value v in System F , $|v|$ may not necessarily be a value. Therefore, given a term t in System F , $|t| \rightarrow^* v_0$ does not necessarily imply that we have $t \rightarrow^* v$ for some value v such that $|v| = v_0$. To have this property, we can impose a restriction on **(T-Tabs)** by requiring that $|t|$ be a value whenever the following rule is applied.

$$\frac{\vec{X}, X; \Gamma \vdash t : T \quad \vec{X} \vdash \Gamma [ctx]}{\vec{X}; \Gamma \vdash \Lambda X.t : \forall X.T} \text{ (T-Tabs)}$$

This restriction is often called value restriction.

Assume $\mathcal{D} :: \emptyset; \emptyset \vdash t : T$ is derivable in System F with value restriction and $|t| \rightarrow u$. Show that $t \rightarrow^* t'$ holds for some term t' such that $|t'| = u$.

Exercise 4 (20 points) The following is a typical implementation of the list reverse function written in ATS:

```
fun{a:t@type} revapp {m,n:nat} .<m>.
  (xs: list (a, m), ys: list (a, n)):<> list (a, m+n) =
  case+ xs of
  | list_cons (x, xs) => revapp (xs, list_cons (x, ys))
  | list_nil () => ys
// end of [revapp]
```

```
fn{a:t@type} reverse {n:nat}
  (xs: list (a, n)):<> list (a, n) = revapp (xs, list_nil ())
// end of [reverse]
```

Please give a paper/pencil proof based on this implementation that the reverse of the reverse of a given list equals the list itself.

Exercise 5 (30 points) Please construct a proof in ATS showing that the reverse of the reverse of a given list equals the list itself. See the file `list.dats` for more details.